Superspinning black holes: theory and practice

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Abstract

Black holes, as described by Einstein’s general theory of relativity, can possess angular momentum, but only a limited amount: too much angular momentum and the event horizon of the black hole disappears, giving rise to a naked singularity, i.e. a singularity not hidden behind an event horizon. The cosmic censorship conjecture states this should never happen. Some theories, however, suggest that there may be supersymmetric objects with a spin higher than the relativistic upper bound. We will take a look at iron fluorescence lines, which have been observed in the spectra emitted by black holes. The profiles of these lines can bear a signature of the relativistic effects of the black hole, which may allow us to measure its spin. The line profiles can be calculated using a computer code, which we then use to try and fit a model to the observed spectrum of the X-ray binary Cygnus X-1.
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Chapter 1

Introduction

The first "scientific" description of gravity was given by Newton\textsuperscript{1} in his magnum opus, the *Philosophiæ Naturalis Principia Mathematica*, published in 1687. Back then, it was unclear whether light should be described as a particle with mass that could be influenced by gravity, as Newton believed, or as a massless wave, as was proposed by Huygens\textsuperscript{2}. It was known already that the speed of light was finite and that objects have an escape velocity. John Michell, in a letter to the Royal Society in 1783, was the first to consider an object that had such a high density that its escape velocity was greater than the speed of light. This would mean that light could not escape, thus making the object invisible. He even went as far as to predict that these objects, which could not be observed directly, might be detectable by their influence on another star orbiting around them. In 1796, Laplace\textsuperscript{3} also got the idea of these invisible stars. However, astronomers at the time were not convinced, especially after the experiments of Thomas Young and later Augustin-Jean Fresnel showed that light behaves as a wave, and thus cannot be affected by gravity. The idea of these invisible stars then faded into obscurity.

In 1915, Einstein\textsuperscript{4} published his new theory of gravity: the general theory of relativity. It is a radically new way to look at gravity: gravitation is no longer a force, but it should instead be seen as a geometric property of spacetime. The metric, a description of the shape of spacetime, dictates the paths of particles. The energy-momentum contents are directly related to the curvature of spacetime, as described by the Einstein field equations. Or to quote John Wheeler: “spacetime tells matter how to move; matter tells spacetime how to curve”. Einstein used

\textsuperscript{1}Isaac Newton, 1643–1727, English physicist, mathematician, astronomer, natural philosopher, alchemist and theologian.

\textsuperscript{2}Christiaan Huygens, 1629–1695, Dutch mathematician, astronomer, physicist, inventor and science fiction writer.

\textsuperscript{3}Pierre-Simon Laplace, 1749–1827, French astronomer and mathematician.

\textsuperscript{4}Albert Einstein, 1879–1955, theoretical physicist.
his new theory to explain the perihelion precession of Mercury’s orbit. He also
made a prediction about the deflection of starlight by the sun, which turned out
to be in good agreement with the measurements done in 1919 by a now famous
expedition led by Arthur Eddington during a solar eclipse, when the moon blocks
the light of the sun, so that the stars near the sun could be observed.

Already in 1915, only a month after Einstein published his theory, Karl
Schwarzschild found the exact solution for a static (i.e. non-rotating) and spheric-
ally symmetric mass. His solution contains two singularities, i.e. points where
the equations blow up and become infinite: one at the centre and one at what
is now called the Schwarzschild radius at $r = 2GM/c^2$, where $M$ is the mass of
the object, $G$ the gravitational constant and $c$ is the speed of light. It was later
realised that singularity at the Schwarzschild surface is not physical, but only a
coordinate singularity: it is an artifact of the coordinates chosen and disappears
after transforming to a different set of coordinates. But it took till the beginning
of the so-called golden age of general relativity (ca. 1960–1980) for scientists to
guess out the true meaning of the Schwarzschild radius. The surface at that
radius functions as an event horizon: nothing, not even light, can escape from
within. Objects of which all matter is within its own event horizon are called
black holes, a term introduced by Wheeler.

Even though Hans Reissner and Gunnar Nordström solved the equations for
a static, spherically symmetric mass with electrical charge in 1918, it was not
until 1963 that Roy Kerr found the exact solutions for a stationary rotating black
hole. These objects have some more bizarre properties. Around a Kerr black hole
there is a region called the ergosphere where it is impossible to remain standing
still, as spacetime itself is dragged along with the rotation of the black hole in a
process called frame-dragging. Within the ergosphere there can be particles with
a negative energy. If these particles cross the event horizon, energy is extracted
from the black hole; this now eponymous process was first thought of by Roger
Penrose in 1969.

At first, the singularity at the centre of a black hole was treated with scepti-
cism, but in the late sixties, Penrose and Stephen Hawking proved the singularity
theorems, which show that once a star has collapsed within its own event horizon,
it inevitably shrinks further and further to become a singularity. The cosmic
censorship conjecture states that in a realistic gravitational collapse, the singular-
ity at the centre of a black hole will always be surrounded by an event horizon.
Naked singularities, i.e. singularities not hidden behind an event horizon, should
not occur in nature. This leads to an upper bound for the amount of angular
momentum a black hole can have. It turns out that in the Kerr solution, the event
horizon disappears if $a > M$, where $a$ is the angular momentum of the black hole
and $M$ its mass. Therefore, these solutions are usually discarded as unphysical.

Interestingly, black holes are actually the simplest objects in our universe.
They are described by only three parameters: their mass, their electrical charge and their angular momentum. Or as John Wheeler put it: "black holes have no hair". Black holes can also be treated as thermodynamical systems. They posses an entropy, which is hard to reconcile with the no-hair theorem, as normally we tend to think of entropy as the logarithm of the number of microstates. Stephen Hawking showed that black holes also radiate, completing the identification of the laws of black hole mechanics with those of thermodynamics: black holes have a temperature and thus emit a thermal spectrum. This leads to the so-called information loss paradox: it appears that all the information that goes into a black hole is lost. Black holes created from different things (chairs or books, tables or cars) all end up as the same black hole. They then slowly evaporate, emitting thermal radiation that contains no information. It seems like the information is lost, a violation of unitarity, a fundamental principle in physics.

Quantum gravity tries to solve the problems that occur when quantum mechanics and general relativity are combined. One popular candidate for quantum gravity is string theory, which claims that particles are not points, but oscillating strings. Thus far, there has been no observational evidence for string theory. In a paper by Gimon and Horava (2009), the suggestion is made that the Kerr bound on the amount of angular momentum a black hole can have in general relativity, can be breached in string theory, so that there might be "superspinning" black hole-like objects. If these objects could be found, they would form a direct observational link to string theory. From supersymmetry comes the so-called BPS bound, which places no constrictions on the amount of angular momentum a black hole can have. It is an exact result, derived from the supersymmetric algebra. The reason the Kerr bound might not be valid is that it is an extrapolation of general relativity to a high-curvature regime: physical laws that hold at low energies may not hold at higher energy scales. The string theoretical objects might resolve the singularity, preventing a naked singularity from occurring. Unfortunately, there are still no precise models for these objects. However, knowledge of the exact details of what happens at the core is not required, as most of the astrophysically interesting phenomena happen at distances of the order of \( r \sim GM/c^2 \), far away from the stringy core.

Of course, black holes do not only exist in theory, but have also been observed. Astrophysical black holes come in two classes: supermassive black holes in the centres of galaxies and stellar mass black holes. The supermassive black holes can have a profound effect on and are closely linked to the formation and evolution of the galaxy they live in. They produce enormous amounts of energy, and their jets function as the most powerful particle accelerators in the universe. They might also halt or start star formation. This thesis mainly deals with the stellar black holes. If these black holes are part of a binary system, matter can be transferred from the companion star to the black hole, see Fig. 1.1. This releases gravitational
potential energy in a process called accretion. Because the matter has angular momentum, it cannot fall to the black hole directly. Instead a disk is formed, where matter slowly spirals in as it loses energy because of viscosity. This energy is then emitted as X-ray radiation, allowing us to see and study the otherwise invisible black hole. The efficiency of the disk is mainly determined by its inner radius. We will see that this radius is directly influenced by the spin of the black hole. The disk can also reflect radiation, leading to emission lines through a process called fluorescence. The shape of these lines is affected by the parameters of the black hole and the disk surrounding it, so they can be used as a probe of the physics going on around the black hole. There is a computer code by Speith et al. (1995) that models these line profiles. We will try to extend this code to also work for superspinning black holes. The code is rather slow, so we will speed it up using parallelisation. Fluorescence line profiles have been observed, but unfortunately, measuring the shape of the line profile is not easy. One needs to model the entire continuum spectrum, before one obtains a profile. In the thesis, we shall describe the efforts taken to model the emitted spectrum of the X-ray binary Cygnus X-1.

We begin Chapter 2 with a thorough introduction to the theoretical aspects of black holes. More information on all of the theoretical concepts mentioned in this introduction can be found in this chapter. It also contains an explanation of
the superspinning black-hole like objects. Chapter 3 deals with the observational aspects of black holes, roughly divided into two parts: first a full treatment of the iron fluorescence line and its profiles as calculated by the Speith code, plus a section on the parallelisation of the code, and then a description of the observations taken of Cygnus X-1 and the models used to explain the data. Finally, we will present our conclusions in Chapter 4. There are three appendices, which further explore some of the concepts mentioned in the theoretical part of the thesis, such as conformal diagrams, quantum field theory in curved spacetime and a derivation of the Hawking radiation.
Chapter 2

Black holes in theory

2.1 Introduction

In this chapter the theoretical side of black holes will be discussed. We will start off with a short introduction to general relativity in Section 2.2. This theory is then used in Section 2.3 to study the way spacetime curves around a black hole, i.e. what the metric looks like for three types of black holes, viz. static black holes, charged black holes and rotating black holes, which includes a discussion of the ergosphere and the Penrose process. The actual theoretical existence of black holes will be established in Section 2.4 by the introduction of singularity theorems. In the same section we will discuss the cosmic censorship conjecture, which is necessary for the derivation of an upper limit on the rotation of black holes. The laws of black hole mechanics will point us into the direction of black hole thermodynamics in Section 2.5. Finally, we will have a look at black holes beyond general relativity in Section 2.6 and the way in which the upper limit on the rotation of black holes can be increased from its general relativistic value.

Throughout this text, we will be using conformal diagrams, which form a nice way of mapping complex spacetimes to a relatively simple diagram. A description and explanation can be found in Appendix A. Some more information on quantum field theory in curved spacetime, necessary for the derivation of the Hawking radiation, can be found in Appendix B. The derivation itself can be found in Appendix C.

2.2 Basics of general relativity

2.2.1 Introduction

This material comes from van Leeuwen (2001).
The general theory of relativity (henceforth: GR) is a theory of gravity. It consists of two parts. On the one hand, there is the description of the curvature of spacetime, which dictates how particles behave. This will be the first part of our discussion. On the other hand, there are the Einstein field equations, which describe how energy-momentum curves spacetime. This will be the second part of our discussion.

GR is built upon the special theory of relativity, which is included in the equivalence principle:

In every point in spacetime $x^\mu$ it is possible to choose a local coordinate system $\xi^\alpha$ in which the laws of physics are the same as in special relativity.

Note that in GR it is not possible to extend our coordinates through the whole of spacetime. In a local coordinate system, gravity is not noticeable and if there are no other forces, object will move in straight lines

$$\frac{d^2 \xi^\alpha}{d\tau^2} = 0 \quad (2.1)$$

In another coordinate system $x^\mu = x^\mu(\xi^\alpha)$, this equation reads

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \quad (2.2)$$

This equation of motion is known as the geodesic equation, where the affine connection is defined as

$$\Gamma^\lambda_{\mu\nu} := \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \quad (2.3)$$

The proper time is defined as

$$d\tau^2 := \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \quad (2.4)$$

where $\eta_{\alpha\beta}$ is the diagonal matrix

$$\eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1) \quad (2.5)$$

It too can be written in general coordinates

$$d\tau^2 := g_{\mu\nu} dx^\mu dx^\nu \quad (2.6)$$

where the metric tensor $g_{\mu\nu}$ is defined as

$$g_{\mu\nu} := \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} \quad (2.7)$$

The Christoffel symbols $\Gamma^\sigma_{\lambda\mu}$ read

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\rho\sigma} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right\} \quad (2.8)$$

and form a connection between $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$. 

2.2.2 Covariant derivatives

The normal derivative of a tensor does not transform as a tensor itself. Therefore, we define the covariant derivative of a contravariant vector \( V^\nu \) as follows

\[
\nabla_\mu V^\nu := \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda
\]  

(2.9)

This combination has the transformational properties of a tensor.

There is now an easy recipe to obtain equations that are valid in GR from ones that are valid in the special theory of relativity:

Write an equation that is valid in the special relativity in tensor form.

Replace \( \eta_{\mu\nu} \) by \( g_{\mu\nu} \) and replace ordinary derivatives by covariant ones.

In a similar way we can define a covariant derivative along a curve \( x^\lambda(s) \) as follows

\[
\frac{DV^\nu}{ds} := \frac{dx^\mu}{ds} \nabla_\mu V^\nu
\]  

(2.10)

Let \( P^\mu \) be a vector defined at the point \( \gamma(s_0) \) on the curve \( \gamma(s) \). Let \( A^\mu(s) \) be a vector obeying

\[
\frac{DA^\mu}{ds} = 0
\]  

(2.11)

Then \( A^\mu(s) \) is called the parallel continuation of \( P^\mu \).

2.2.3 Curvature

Define the Riemann-Christoffel curvature tensor as follows

\[
R^\lambda_{\mu\kappa\nu} := \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} + \Gamma^\eta_{\mu\kappa} \Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\kappa\eta}
\]  

(2.12)

If we parallel transport a vector round a sufficiently small curve around some point \( X \) it will only stay the same if the curvature \( R^\lambda_{\mu\kappa\nu} \) is zero in \( X \). The commutator of the covariant derivative is given by

\[
[\nabla_\nu, \nabla_\kappa]V_\mu = V_\sigma R^\sigma_{\mu\kappa\nu}
\]  

(2.13)

That is to say, the covariant derivative only commutes when the curvature is zero.

By taking the contraction between the first and the third index of the Riemann tensor, we obtain the Ricci tensor

\[
R_{\mu\nu} := g^{\lambda\kappa} R^\lambda_{\mu\kappa\nu}
\]  

(2.14)

When taking two contractions, we get

\[
R := g^{\lambda\kappa} g^{\mu\nu} R_{\lambda\kappa\mu\nu}
\]  

(2.15)

a scalar.
2.2.4 The Einstein field equations

We have now seen how to describe the curvature of spacetime. It is now time for the last part of our discussion, viz. the way in which energy-momentum curves spacetime.

The energy-momentum tensor is a generalisation of the $p^\mu$ four-vector in special relativity.

$$T^{\alpha\beta}(x) = \sum_n p_n^\alpha p_n^\beta \delta^{(3)}(x - x_n(t))$$  \hspace{1cm} (2.16)

The Einstein field equations read

$$R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$$  \hspace{1cm} (2.17)

This equation clearly forms a link between the energy-momentum contents and the curvature of spacetime.

It is not possible to derive the Einstein field equations from first principles, but there is the Hilbert action

$$S_H = \int \sqrt{-g} R d^4x$$  \hspace{1cm} (2.18)

Finding stationary points of $S_H$ under variations of the metric leads to Eqn. 2.17.

2.2.5 The Newtonian limit

In this section, we will try to find out if and how the general theory of relativity encapsulates the Newtonian theory of gravity.

Consider a particle moving slowly in a weak stationary gravitational field. First of all, if the particle moves slow enough, we can neglect the $dx^i/d\tau$ compared to the $dt/d\tau$, so the geodesic equation (Eqn. 2.2) becomes

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{00} \left( \frac{dt}{d\tau} \right)^2 = 0$$  \hspace{1cm} (2.19)

Secondly, a static field means $\partial_0 g_{\mu\nu} = 0$, so the Christoffel symbols (Eqn. 2.8) become

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\nu} \left\{ \frac{\partial g_{0\nu}}{\partial x^0} + \frac{\partial g_{0\nu}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\nu} \right\}$$

$$= -\frac{1}{2} g^{\mu\nu} \frac{\partial g_{00}}{\partial x^\nu}$$  \hspace{1cm} (2.20)
Finally, the fact that the gravitational field is weak means that we can decompose the metric tensor $g_{\mu\nu}$ into the Minkowski part $\eta_{\mu\nu}$ and a small perturbation $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1 \quad (2.21)$$

Now, filling in Eqn. 2.21 into Eqn. 2.20, we find to first order in $h_{\mu\nu}$

$$\Gamma^\mu_{\nu\rho} = -\frac{1}{2} \eta^{\mu\nu} \frac{\partial h_{\rho00}}{\partial x^\nu} \quad (2.22)$$

So the geodesic (Eqn. 2.19) becomes

$$\frac{d^2 x^\mu}{d\tau^2} - \frac{1}{2} \eta^{\mu\nu} \frac{\partial h_{00}}{\partial x^\nu} \left(\frac{dt}{d\tau}\right)^2 = 0 \quad (2.23)$$

Explicitly writing down the time and space components

$$\frac{d^2 t}{d\tau^2} = 0$$

$$\frac{d^2 x^i}{d\tau^2} = \frac{1}{2} \frac{\partial h_{00}}{\partial x^i} \left(\frac{dt}{d\tau}\right)^2 \quad (2.24)$$

So $dt/d\tau$ is static and we can rewrite the spacelike components as follows

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} \frac{\partial h_{00}}{\partial x^i} \quad (2.25)$$

The corresponding Newtonian result reads

$$\frac{d^2 x^i}{dt^2} = -\frac{\partial \phi}{\partial x^i} \quad (2.26)$$

where $\phi$ is the gravitational potential. The two are equal if we identify

$$h_{00} = -2\phi \quad (2.27)$$

The metric (Eqn. 2.21) then becomes

$$g_{00} = -(1 + 2\phi) \quad (2.28)$$

So we see that the Newtonian formulation integrates nicely into the relativistic framework. For a mass $M$ the potential at distance $r$ is given by

$$\phi = -\frac{GM}{r} \quad (2.29)$$

where $G$ is the gravitational constant.
2.3 Black hole solutions in general relativity

In this section, we will take a look at the black hole solutions in general relativity. The material is taken from the chapters on black holes in Carroll (2004) and the lecture notes by Townsend (n.d.).

2.3.1 Spherical symmetry and the Schwarzschild metric

In this section, we will find the solution for the simplest black hole, the Schwarzschild metric. Before we discuss this metric, let us first take a closer look at spherical symmetry. We have seen that the metric is defined as:

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \] (2.30)

For Minkowski space, this reads

\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \] (2.31)

or, changing to polar coordinates,

\[ ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \] (2.32)

where

\[ d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \] (2.33)

is the metric on a unit two-sphere. Here, the angles \( \theta \) and \( \phi \) have the following ranges

\[ \begin{align*}
\theta & \in [0, \pi] \\
\phi & \in [0, 2\pi)
\end{align*} \] (2.34)

We are free to multiply all the terms of Eqn. 2.32 by separate prefactors

\[ ds^2 = -e^{\alpha(r)}dt^2 + e^{\beta(r)}dr^2 + e^{\gamma(r)}r^2d\Omega^2 \] (2.35)

as the shape of the metric stays the same, i.e. the coefficient of the \( d\phi^2 \) is still \( \sin^2 \theta \) that of the \( d\theta^2 \) term. We have used exponentials so that the sign of the terms stays the same. Define a new coordinate \( \bar{r} \) as

\[ \bar{r} = e^{\gamma(r)}r \] (2.36)

which has the following differential

\[ d\bar{r} = e^\gamma dr + e^\gamma r d\gamma = \left(1 + r \frac{d\gamma}{dr}\right) e^\gamma dr \] (2.37)
In terms of the new radial coordinate, Eqn. 2.35 becomes
\[ ds^2 = -e^{2\alpha(r)} dt^2 + \left( 1 + r \frac{d\gamma}{dr} \right)^{-2} e^{2\beta(r)-2\gamma(r)} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2 \] (2.38)

Now make the following relabellings
\[ \tilde{r} \rightarrow r \quad \left( 1 + r \frac{d\gamma}{dr} \right)^{-2} e^{2\beta(r)-2\gamma(r)} \rightarrow e^{2\beta(r)} \] (2.39)

The metric (Eqn. 2.38) then becomes
\[ ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2 \] (2.40)

which is the same as Eqn. 2.35 expect for the $\gamma$ factor. Note that we have not chosen $e^{2\gamma(r)}$ to be one, as that would be a statement about the geometry of our system. Instead, we have chosen coordinates such that this factor simply does not exist. One could ask what the meaning is of our new coordinates. According to van Leeuwen (2001), we identify our coordinates to real world measurable quantities at the end of our calculation. In this case, it turns out that the $r$ coordinate corresponds to our notion of radial distance, viz. something you can measure with a ruler. Eqn. 2.40 is the standard form of the static isotropical metric.

We can now proceed to the Schwarzschild metric. Start with the Minkowski metric in case of a static and spherically symmetric space, Eqn. 2.40. Calculate the Christoffel symbols, the Riemann tensor and finally the Ricci tensor. The Einstein field equations (see Eqn. 2.17) in vacuum read
\[ R_{\mu\nu} = 0 \] (2.41)

Use this to determine $\alpha$ and $\beta$. We obtain the Schwarzschild metric
\[ ds^2 = -\left( 1 - \frac{R_S}{r} \right) dt^2 + \left( 1 - \frac{R_S}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 \] (2.42)

The constant $R_S$, the Schwarzschild radius, can be found by comparing the metric to the weak-field limit, in which the $tt$ component reads (see Eqn. 2.28)
\[ g_{00} = -\left( 1 - \frac{2GM}{r} \right) \] (2.43)

The two are the same if we identify
\[ R_S = 2GM \] (2.44)
Figure 2.1: Schwarzschild spacetime using the \((v, r)\) coordinates.

We interpret \(M\) as the mass. Note that as \(M \to 0\), we recover the Minkowski metric. The same happens for \(r \to \infty\), a property known as asymptotic flatness.

We see that there is a singularity at \(r = 0\). At \(r = 2GM\) there is no actual singularity, but the problem is caused by a bad coordinate system. It is therefore sometimes referred to as a coordinate singularity. Transform to more appropriate coordinates by defining the tortoise coordinate \(r^*\) as

\[
r^* = r + 2GM \ln \left( \frac{r}{2GM} - 1 \right)
\]  

(2.45)

Next define

\[
\begin{align*}
v &= t + r^* \\
u &= t - r^*
\end{align*}
\]  

(2.46)

The combination of the spacelike coordinate \(r\) and the timelike coordinate \(v\) are known as the Eddington-Finkelstein coordinates, in terms of which the metric reads

\[
ds^2 = - \left( 1 - \frac{2GM}{r} \right) dv^2 + (dvdr + drdv) + r^2 d\Omega^2
\]  

(2.47)

A diagram using the new coordinates is drawn in Fig. 2.1. We see that at \(r = 2GM\) we are not dealing with a singularity, but with an event horizon. The light cones tilt over, so that for \(r < 2GM\), all future-directed paths are in the direction of decreasing \(r\). If the matter is inside its own event horizon, we have a so-called black hole. In general, the event horizon can also be found by \(g^{rr} = 0\).
Now we want to find an appropriate conformal transformation, as explained in Appendix A. Define

\[
\begin{align*}
v' &= e^{v/4GM} \\
u' &= -e^{-u/4GM}
\end{align*}
\] (2.48)

In terms of these coordinates, the Schwarzschild metric reads

\[
ds^2 = -\frac{16G^3M^3}{r}e^{-r/2GM} (dv'du' + du'dv') + r^2d\Omega^2
\] (2.49)

where \( r \) is defined implicitly via

\[
v' u' = -\left( \frac{r}{2GM} - 1 \right) e^{r/2GM}
\] (2.50)

Again, just as in flat spacetime, we use the arctan to map infinity to a finite coordinate value

\[
\begin{align*}
v'' &= \arctan \left( \frac{v'}{\sqrt{2GM}} \right) \\
u'' &= \arctan \left( \frac{u'}{\sqrt{2GM}} \right)
\end{align*}
\] (2.51)

with ranges

\[
\begin{align*}
-\frac{\pi}{2} < v'' < +\frac{\pi}{2} \\
-\frac{\pi}{2} < u'' < +\frac{\pi}{2} \\
-\frac{\pi}{2} < v'' + u'' < +\frac{\pi}{2}
\end{align*}
\] (2.52)

The \( v'', u'' \) metric is conformally related to Minkowski space. The conformal diagram for Schwarzschild spacetime is drawn in Fig. 2.2. The singularity is indicated by a wavy line. We see that once you cross the line \( r = 2GM \) there is no escape, as all timelike paths bring you to \( r = 0 \). Notice that the structure of conformal infinity is equal to that of Minkowski space, as it should be because of asymptotic flatness.

### 2.3.2 Adding charge

In the section we present the exact solutions for electrically charged black holes. Again, we have a spherically symmetric metric, see Eqn. 2.40. In this case however,
we are no longer dealing with a vacuum. The black hole has a electromagnetic field, which will act as a source of energy-momentum

\[ T_{\mu\nu} = F_{\mu\rho} F^{\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \]  \hspace{1cm} (2.53)

where \( F_{\mu\nu} \) is the electromagnetic field strength tensor. As we are considering the spherical symmetry, the most general case is given by

\[ F_{tr} = f(r, t) = -F_{rt} \]
\[ F_{\theta\phi} = g(r, t) \sin \theta = -F_{\phi\theta} \]  \hspace{1cm} (2.54)

The field equations in this case are both the Maxwell equations

\[ g^{\mu\nu} \nabla_{\mu} F_{\nu\sigma} = 0 \]
\[ \nabla_{[\mu} F_{\nu\rho]} = 0 \]  \hspace{1cm} (2.55)

and the Einstein field equations (see Eqn. 2.17)

\[ R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) \]  \hspace{1cm} (2.56)

The two are coupled, since both depend on the metric and the electromagnetic field strength tensor. The set of equations can be solved however, to yield the Reissner-Nordström metric

\[ ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2 \]  \hspace{1cm} (2.57)

where

\[ \Delta = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2} \]  \hspace{1cm} (2.58)
Here, $M$ is interpreted as the mass and $Q$ and $P$ are the electric and magnetic charge respectively (though the latter probably has to be taken cum grano salis).

There is a true singularity at $r = 0$. Note that it is a timelike line, not a spacelike surface as in Schwarzschild, as can be seen by determining the sign of $ds^2$. The event horizon can be found by setting $g^{rr}$ to zero

$$g^{rr}(r) = \Delta(r) = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2} = 0$$

(2.59)

with solution

$$r_{\pm} = GM \pm \sqrt{G^2 M^2 - G(Q^2 + P^2)}$$

(2.60)

We can distinguish between three different cases.

If $GM^2 < Q^2 + P^2$, we have no event horizon, but we do have a singularity at $r = 0$, which violates the cosmic censorship conjecture (see the next section). The conformal diagram, given in Fig. 2.3, has a singularity on the left at $r = 0$.

If $GM^2 > Q^2 + P^2$, we have two event horizons at $r = r_{\pm}$. If you cross $r_+$, the coordinate $r$ becomes a timelike coordinate and you are forced to move in the direction of decreasing $r$. But when you reach $r_-$, $r$ becomes a spacelike coordinate again. You can now choose to carry on moving to the singularity at $r = 0$ or you can choose to go back again in the direction of increasing $r$, back through $r = r_-$. Here $r$ becomes timelike again, but this time in the opposite direction: you have no choice but to move in the direction of increasing $r$ and through $r = r_+$. This can be seen in the conformal diagram in Fig. 2.4.

The final case is the extreme solution: $GM^2 = Q^2 + P^2$. This configuration is of course unstable, as even one infalling particle will disturb the equilibrium.
Figure 2.4: The conformal diagram for the Reissner-Nordström solution with $GM^2 > Q^2 + P^2$. Figure taken from Carroll (2004).
Even though there is an “event horizon” at \( r = GM \), the \( r \) coordinate never becomes timelike. The conformal diagram is shown in Fig. 2.5.

**Electrons as black holes**

One could wonder how the ratio of charge and mass of an electron compares to a black hole. First of all, we will reed to reintroduce the numerical constants we left out in the beginning. If we do that, the extreme solution \( GM^2 = Q^2 \) (ignoring magnetic charges) is given by

\[
\frac{Q^2}{M^2} = G4\pi\epsilon_0 = 7.4 \cdot 10^{-21} \text{s}^2 \text{A}^2/\text{kg}^2 \quad (2.61)
\]

For an electron, we have

\[
\frac{Q_e^2}{M_e^2} = 3.1 \cdot 10^{22} \text{s}^2 \text{A}^2/\text{kg}^2 \quad (2.62)
\]

a difference of 43 orders of magnitude. Of course, we cannot apply relativity in this regime, as quantum effects dominate on this scale. Still, the calculation is interesting, as it shows that we can, at least theoretically, create an extremal black hole by shooting enough electrons into a black hole.
2.3.3 Adding angular momentum

In this section, we consider a rotating (electrically neutral) black hole. This case is no longer static, but stationary, and also no longer spherically symmetric, but axial symmetric. The solution is known as the Kerr metric

\[ ds^2 = - \left( 1 - \frac{2GMr}{\rho^2} \right) dt^2 - \frac{2GMar \sin^2 \theta}{\rho^2} (dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} \left[ (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \right] d\phi^2 \] (2.63)

where

\[ \Delta(r) = r^2 - 2GMr + a^2 \] (2.64)

and

\[ \rho^2(r, \theta) = r^2 + a^2 \cos^2 \theta \] (2.65)

Here \( M \) is the mass and \( a \) is the angular momentum per unit mass

\[ a = J/M \] (2.66)

As \( a \to 0 \) we retrieve the Schwarzschild metric.

The singularity occurs when \( \rho = 0 \) or

\[ \rho^2 = r^2 + a^2 \cos^2 \theta = 0 \] (2.67)

We see that both the \( r \) and the cosine term need to be zero

\[ r = 0, \quad \theta = \pi/2 \] (2.68)

so the singularity is not a point, but a ring. (In this case, \( r \) is not the radial coordinate of a polar coordinate system, but part of the so-called Boyer-Lindquist coordinates.)

The event horizons occur at those values of \( r \) for which \( g^{rr} = 0 \). Since \( g^{rr} = \Delta/\rho^2 \) and \( \rho^2 \geq 0 \) this happens when

\[ \Delta(r) = r^2 - 2GMr + a^2 = 0 \] (2.69)

with solution

\[ r_{\pm} = GM \pm \sqrt{G^2M^2 - a^2} \] (2.70)

Just like in the Reissner-Nordström case we have three types of solutions: \( GM < a \), \( GM = a \) and \( GM > a \). The first one features a naked singularity and the second (extremal) case is unstable, just like in Reissner-Nordström: one infalling particle will disturb the delicate equilibrium. We therefore focus on the third case, for which the conformal diagram is given in Fig. 2.6.
Figure 2.6: The conformal diagram for the Kerr solution with $GM > a^2$. Figure taken from Carroll (2004).
The ergosphere

The stationary limit surface can be found by solving

$$K^\mu K_\mu = g_{tt} = -\frac{1}{\rho^2} (\Delta - a^2 \sin^2 \theta) = 0$$

where $K = \partial_t$ is the time-translation Killing vector. We find

$$(r - GM)^2 = G^2 M^2 - a^2 \cos^2 \theta$$

(2.72)

Note the dependence on the angle $\theta$. If we compare this to the outer event horizon as given by Eqn. 2.70

$$(r_+ - GM)^2 = G^2 M^2 - a^2$$

(2.73)

we see that there is a region between the two surfaces, the so-called ergosphere, see Fig. 2.7. Inside the ergosphere you must move along in the direction of the rotation of the black hole. This can be seen by considering a photon emitted in the $\phi$ direction, so $dr = d\theta = 0$. We then have

$$ds^2 = 0 = g_{tt} dt^2 + g_{t\phi}(dt d\phi + d\phi dt) + g_{\phi\phi} d\phi^2$$

(2.74)

Solving for $d\phi/dt$ and using the quadratic formula, we obtain

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

(2.75)
Evaluating at the stationary limit surface, where \( g_{tt} = 0 \), we have

\[
\frac{d\phi}{dt} = 0, \quad \frac{d\phi}{dt} = -\frac{2g_{t\phi}}{g_{\phi\phi}} = \frac{a}{2G^2M^2 + a^2}
\]

We see that light either moves along with the rotation of the black hole, or it stands still. Massive particles move more slowly than light and thus are dragged along in a process called frame-dragging. The minimum value of the angular velocity of a particle at the outer event horizon is given by

\[
\Omega_H = \left( \frac{d\phi}{dt} \right)_{r_+} = \frac{a}{r_+^2 + a^2}
\]

This quantity can be thought of as a measure of the angular velocity of the black hole.

The Penrose process

We shall make use of the conserved quantities associated with the Killing vectors \( K = \partial_t \) and \( R = \partial_\phi \). The four-momentum is given by

\[
p^\mu = m \frac{dx^\mu}{d\tau}
\]

where \( m \) is the rest mass of a particle. The energy and angular momentum of the particle are given by

\[
E = -K_\mu p^\mu = m \left( 1 - \frac{2GMr}{\rho^2} \right) \frac{dt}{d\tau} + \frac{2mGMar}{\rho^2} \sin^2 \theta \frac{d\phi}{d\tau}
\]

and

\[
L = R_\mu p^\mu = -\frac{2mGMar}{\rho^2} \sin^2 \theta \frac{dt}{d\tau} + \frac{m(r^2 + a^2)^2 - m\Delta a^2 \sin^2 \theta}{\rho^2} \sin^2 \theta \frac{d\phi}{d\tau}
\]

Note the minus sign in the definition of the energy. This is because both \( K^\mu \) and \( p^\mu \) are timelike (at infinity), so their inner product is negative, but we want the energy to be positive. Inside the ergosphere, however, \( K^\mu \) becomes spacelike, which means that the energy of the particle can be negative. This leads to an interesting way to extract energy from the black hole. Suppose we enter, armed with a stone, the black hole with total energy \( E(0) \). If we now throw the stone in such a way that the energy of the stone is negative (\( E(2) < 0 \)) and we leave the black hole again with energy \( E(1) \), we have because of conservation of energy that

\[
E(0) = E(1) + E(2)
\]
Hence $E^{(1)} > E^{(2)}$ and we have extracted energy from the black hole. Penrose showed that the now eponymous process is actually possible. Now consider the following linear combination of the time-translation and the rotational Killing vectors

$$\chi^\mu = K^\mu + \Omega_H R^\mu$$

where $\Omega_H$ is defined in Eqn. 2.77. This combination is null at the outer event horizon. For the rock to be crossing the event horizon forward in time we must have

$$p^{(2)\mu} \chi^\mu < 0$$

or

$$L^{(2)} < \frac{E^{(2)}}{\Omega_H}$$

The angular momentum is negative and hence moving against the direction of rotation. The mass and the angular momentum of the black hole change accordingly

$$\delta M = E^{(2)}$$

$$\delta J = L^{(2)}$$

We can now write Eqn. 2.84 to become a limit for the amount of angular momentum we can extract from the black hole

$$\delta J < \frac{\delta M}{\Omega_H}$$

### 2.4 Singularity theorems and the cosmic censorship conjecture

The singularity theorems of Penrose and Hawking show that after gravitational collapse we end up with a singularity. The theorems are important, as they establish the actual existence of black holes. Hawking and Ellis (1973) treat these theorems in a rather mathematical way.

First of all, to be able to define singularities in spacetime, we need the idea of geodesic completeness, which means that every geodesic can be extended to arbitrary values of its affine parameter. We distinguish between three kinds of geodesic incompleteness: timelike, null and spacelike geodesics. These do not necessarily coincide. If a spacetime is timelike or null geodesically incomplete, it has a singularity.

The first theorem about a singularity was given by Penrose in 1965. He showed that once a star passes inside the Schwarzschild radius, there will be a closed trapped surface $\mathcal{T}$. This is a surface such that two null geodesics
orthogonal to $\mathcal{T}$ are converging at $\mathcal{T}$. Or in other words, even the outgoing light rays are dragged back and are in fact converging. Matter within $\mathcal{T}$ is trapped in a succession of surfaces of smaller and smaller area. Penrose proved that this inevitably leads to a singularity.

There is however a problem with this theorem: it does not tell us if we end up with a singularity or a Cauchy surface. A further theorem by Hawking and Penrose (1970) establishes the existence of singularities under very general conditions, but does not show whether the singularity is in the past or future. This is solved by another theorem of Hawking (1967), which has as a drawback that there may be a closed timelike curve instead of a singularity. Yet another theorem by Hawking (1967) fixes this problem.

The weak cosmic censorship conjecture by Penrose (1969) tells us that naked singularities, i.e. a singularity not hidden behind an event horizon, so that light signals from it can reach $\mathcal{J}^+$, cannot form in gravitational collapse from a generic, initially non-singular state in an asymptotically flat spacetime. A spacetime is asymptotically flat if it resembles Minkowski at large distances and if the curvature goes to zero at infinity. Wald (1997) gives a more precise definition of the conjecture.

It is a conjecture, as no proof has been found. There are however strong indications that the conjecture is correct.

First of all, there is the stability of black holes. If weak cosmic censorship would not exist, gravitational collapse would not always lead to a black hole, but could also lead to a naked singularity. If one studies linear perturbation theory on a background spacetime containing a black hole, one could try to look for signs of boundless growth, which would signal that the black hole became a naked singularity. Cosmic censorship was demonstrated to hold for several cases.

Next, there is the failure to produce counterexamples. Possible counterexamples were proven to actually obey cosmic censorship, see Wald (1997).

Finally, there is the proof of a cosmic censorship theorem for a non-trivial, special case, viz. the spherically symmetric Einstein-Klein-Gordon system, the details of which are again given in Wald (1997).

The cosmic censorship conjecture important, as it implies that there is an upper limit on the spin of a Kerr black hole

$$a < GM$$

as above this limit we would have a naked singularity.
2.5 Black hole mechanics and thermodynamics

The area theorem states that the area of the event horizon is non-decreasing. The idea behind it is that even though the mass decreases, the area still increases because the angular momentum also decreases. We check the theorem by calculating the area of the outer event horizon. The induced metric $\gamma_{ij}$ on the horizon can then be found by putting $r = r_+$ (so $\Delta = 0$, $dt = dr = 0$ in the Kerr metric (Eqn. 2.63)

$$\gamma_{ij}dx^idx^j = ds^2(dt = 0, dr = 0, r = r_+) = (r_+^2 + a^2 \cos^2 \theta)d\theta^2 + \left[\frac{(r_+^2 + a^2)^2 \sin^2 \theta}{r_+^2 + a^2 \cos^2 \theta}\right] d\phi^2 \quad (2.89)$$

The horizon area can then be found by integrating the induced volume element

$$A = \int \sqrt{|\gamma|}d\theta d\phi \quad (2.90)$$

The determinant is easily found

$$|\gamma| = (r_+^2 + a^2)^2 \sin^2 \theta \quad (2.91)$$

so that the horizon area becomes

$$A = 4\pi (r_+^2 + a^2) \quad (2.92)$$

The irreducible mass of the black hole is defined as

$$M^{2\text{irr}} = \frac{A}{16\pi G^2} = \frac{1}{2} \left( M^2 + \sqrt{M^4 - (J/G)^2} \right) \quad (2.93)$$

with the following differential

$$\delta M^{2\text{irr}} = \frac{a}{4GM^{\text{irr}} \sqrt{G^2 M^2 - a^2}} (\Omega_H^{-1} \delta M - \delta J) \quad (2.94)$$

so because of Eqn. 2.87

$$\delta M^{\text{irr}} > 0 \quad (2.95)$$

So indeed the irreducible mass can never be reduced. Because of this, the area can never decrease either. Using Eqn. 2.93 and 2.94, we can now write

$$\delta A = 8\pi G \frac{a}{\Omega_H \sqrt{G^2 M^2 - a^2}} (\delta M - \Omega_H \delta J) \quad (2.96)$$

or

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega_H \delta J \quad (2.97)$$
where the surface gravity $\kappa$ is given by

$$\kappa = \frac{\sqrt{G^2 M^2 - a^2}}{2GM (GM + \sqrt{G^2 M^2 - a^2})} \quad (2.98)$$

It turns out that there is a close connection between black holes and thermodynamics. Consider the first law of thermodynamics

$$dE = TdS - pdV \quad (2.99)$$

where $T$ is the temperature, $S$ is the entropy, $p$ is the pressure and $V$ is the volume, so the first term on the right hand side is the heat we add to the system and the second term is the work we do to the system. If we compare this equation to Eqn. 2.97, we see that the two correspond if we make the following identifications:

$$E \leftrightarrow M \quad (2.100)$$
$$S \leftrightarrow A/4G \quad (2.101)$$
$$T \leftrightarrow \kappa/2\pi \quad (2.102)$$

Furthermore, the second law of thermodynamics, which tells us that the entropy never decreases, is analogous to the area theorem.

If these identifications are correct, it would mean that black holes have a temperature. This is quite surprising, as it would imply that black holes are in fact not totally black, but actually emit radiation. Stephen Hawking was the first to calculate the now eponymous temperature of black holes. A derivation is not given here, but in Appendix C, as it is rather lengthy and not required for the rest of our discussion. The Hawking temperature concludes the identification of the black hole and thermodynamical quantities.

The entropy turns out to be huge. The total entropy of the visible universe is of the order of $10^{88}$, while the black hole entropy (in astrophysical units) is given by

$$S_{BH} \approx 10^{90} \left( \frac{M}{10^6 M_\odot} \right)^2 \quad (2.103)$$

We see that the entropy of one super-massive black hole is larger than the entropy of the entire universe. This brings us to the problem of the interpretation of this entropy. Normally, we would like to define the entropy as the logarithm of the number of states. However, the so-called no-hair theorem states that a black hole is completely defined by its mass, spin and charge.

This problem could possibly be solved by assuming that all the information about the state of a black hole would be conveniently hidden behind its event
horizon. However, the Hawking effect radiates away energy and hence the black hole evaporates. As more and more mass is lost, the surface gravity increases and with it the temperature. Smaller black holes radiate more than bigger ones. Hence, a black hole has a finite lifetime of the order of

\[ \tau_{BH} = \left( \frac{M}{M_\odot} \right)^3 \times 10^{71} \text{s} \]  

(2.104)

This number is absolutely huge if we realise that the Hubble time is of the order of \( H_0^{-1} \approx 10^{18} \text{s} \). Still, even though it takes a very long time, eventually the black hole will evaporate, which means that we cannot hide states behind the event horizon. Since Hawking radiation is thermal, it cannot contain any information. When the black hole has disappeared, all we are left with is this information-less radiation. This means that two black holes will produce the same radiation, regardless of how they were created. This gives rise to the so-called information loss paradox. This is a problem, as any quantum physical theory should be unitary, i.e. the information required to specify the initial state should be the same as that needed to specify the final state.

2.6 Beyond general relativity

2.6.1 Introduction

In the previous sections we have seen that in general relativity (GR) there is a limit on the rotation of black holes, the so-called Kerr bound, if we assume the cosmic censorship conjecture, or the absence of naked singularities, to hold

\[ a^2 + Q^2 \leq M^2 \]  

(2.105)

From supersymmetry comes the BPS bound (Gimon and Hořava (2009))

\[ Q^2 \leq M^2 \]  

(2.106)

This bound is less restrictive, but stronger than the GR bound, as there are no assumptions made.

2.6.2 BPS bound

First of all, let us take a closer look at the BPS bound. The following is based on Mohaupt (2000) and Lykken (1996). For an introduction to supersymmetry, see de Roo (1995).
The Coleman-Mandula theorem states that there is a fixed number of symmetries of the S-matrix. However, this theorem assumes that the symmetry algebra consists only of commutators. Also allowing anti-commuting generators, one obtains supersymmetry (SUSY). Haag, Lopuzanski and Sohnius showed that supersymmetry is the only possible additional symmetry of the S-matrix.

Consider a supersymmetric theory, i.e. a theory with conserved spinorial currents. If there are $N$ of these, we get $4N$ real conserved charges. We organise these into $N$ Weyl spinors $Q^A$, which are the supersymmetry generators. Here, the upper index $A = 1, \ldots, N$ counts the supersymmetries and $\alpha = 1, 2$ is a Weyl spinor index. The theorem of Haag, Lopuzanski and Sohnius states that the most general supersymmetry algebra (in four spacetime dimensions) is

$$\{Q^A, Q^B\} = 2\sigma_{\alpha\beta} \delta^{AB}$$
$$\{Q^A, Q^B\} = 2\epsilon_{\alpha\beta} Z^{AB}$$

(2.107)

Suppose that $N = 2$. We want to construct massive representations $M^2 > 0$. The momentum operator $P_\mu$ can then be brought to the standard form $P_\mu = (-M, 0)$. Plugging this into the algebra (Eqn. 2.107) and setting $2|Z| = |Z^{12}|$, the algebra takes the following form

$$\{Q^A, Q^B\} = 2M \delta_{\alpha\beta} \delta^{AB}$$
$$\{Q^A, Q^B\} = 2|Z| \epsilon_{\alpha\beta} \epsilon^{AB}$$

(2.108)

Now rewrite the algebra in terms of fermionic creation and annihilation operators. By taking appropriate linear combinations of the supersymmetry charges one can bring the algebra to the form

$$\{a_{\alpha}, a^\dagger_{\beta}\} = 2(M + |Z|) \delta_{\alpha\beta}$$
$$\{b_{\alpha}, b^\dagger_{\beta}\} = 2(M - |Z|) \delta_{\alpha\beta}$$

(2.109)

Now we can choose any irreducible representation $|s\rangle$ of the little group $SO(3)$ of massive particles and take the $a_{\alpha}, b_{\beta}$ to be annihilation operators,

$$a_{\alpha} |s\rangle = 0, \quad b_{\beta} |s\rangle = 0$$

(2.110)

Then the basis of the corresponding irreducible representation of the super Poincaré algebra is

$$\mathcal{B} = \{a_{\alpha_1}^\dagger \ldots b_{\beta_1}^\dagger \ldots |s\rangle\}$$

(2.111)

In the context of quantum mechanics we are only interested in unitary representations. Therefore we have to require the absence of negative norm states. This implies that the mass is bounded by the central charge

$$M \geq |Z|$$

(2.112)

This is the so-called BPS-bound. It is an exact bound, as it derives from the supersymmetric algebra.
2.6.3 Interpretation

If the only bound is the BPS one, we see that there is actually no limit on the angular momentum a black hole can have. But why would the GR bound be broken? According to Gimon and Hořava (2009), the way to look at this is as follows. By using GR in the high-curvature regime, we are extrapolating, which is a risky thing to do. Effects on a given energy scale do not need to hold on a different energy scale.

Are there actual examples of theories that break the GR bound? There are several solutions, each of which add high-energy modifications to GR at the core. The question remains how physical these are are.

In 4+1 dimensions there are the BMPV black holes. From large distances, the heterotic string theory solutions look 3+1 dimensional and appear to contain a naked over-rotating singularity. However, according to the string solution, their core is resolved by Kaluza-Klein modes of two extra dimensions compactified on a torus: instead of a naked singularity, the core contains a periodic array of black holes along the compact dimensions. In this way, an object that is legitimate in string theory can appear to be a Kerr bound violating black hole.

Gimon and Hořava (2009) also give another example: reduce black ring solutions with residual angular momentum. These reductions typically have a multi-centre structure and extra U(1) fields which drive angular momentum up via dipole moments. Outside of SUSY, there exist extremal Kaluza-Klein black holes with a slow rotation phase and a fast rotation phase separated by a singular configuration.
Chapter 3

Black holes in practice

3.1 Introduction

In Chapter 2, we have explored the theoretical background of black holes. This chapter will contain the astrophysical part of the thesis. We will start by introducing the two types of astrophysical black holes, followed by an explanation of accretion theory. This includes studying the behaviour of particles around a black hole, both for the static and the rotating case. Crucial will be the derivation of the ISCO, the innermost stable circular orbit. It directly influences the accretion efficiency of the black hole. We then proceed with an introduction to the iron fluorescence line. In the next section, we will see how this line is affected by the gravitational effects of the rotating black hole. For this, we will be using a code that is able to generate a line profile for any value of the black hole spin parameter. Both the code and the theory behind it will be explained and a description will be given of the efforts taken to extend this code for usage with superspinning black holes, i.e. black holes with a spin parameter greater than one. Finally, we will describe the modelling of Cygnus X-1 spectral data. It will turn out to be pivotal to properly describe the entire continuum spectrum, as this determines the shape of the observed iron line.

3.2 Astrophysical black holes

Astrophysical black holes come in two distinct classes. For a review, see Reynolds and Nowak (2003).

First, there are the stellar mass black holes, which arise from the collapse of a massive star and form a class called the galactic black hole candidates (GBHCs). Members of this class are compact objects that are more massive than a neutron
star can be¹. The first black hole candidate to be found this way was the X-ray binary Cygnus X-1, which has a mass of approximately $10 \, \text{M}_\odot$, far above the maximum neutron star mass. This source was detected in 1964 by an X-ray detector carried on a rocket. Later, the X-ray emitter was identified with its optical companion star, which allowed for its mass determination². As we will discuss in the next section, the source’s enormous X-ray luminosity is powered by the accretion of matter from the companion star onto the black hole. After the discovery of Cygnus X-1, more than a score of these objects have been found.

A second category is comprised of supermassive black holes (SMBHs), which reside in the centre of galaxies and have masses of $10^5$–$10^9 \, \text{M}_\odot$. Even though the centre of our galaxy is relatively nearby (only 8 kpc away), the view in the optical band of the spectrum is blocked by the large amounts of dust in the galactic plane. However, by studying stars in the centre of our galaxy in the near-infrared using high-resolution (down to 300 μas) telescopes for 16 years, it has been found that in the centre of our galaxy there has to be an invisible mass of $4.31 \cdot 10^6 \, \text{M}_\odot$ (see Gillessen et al. (2009)) within a radius of 0.58 mpc (see Ghez et al. (2008)) to explain their trajectories. As any other concentration of matter this massive and compact would have collapsed on a timescale far shorter than the age of the Milky Way, this effectively rules out anything but a black hole (see Doeleman et al. (2008))³.

SMBHs have also been found in other galaxies. By measuring the velocity profile of the inner 20 pc of the gas disk of M87 with the Hubble Space Telescope (HST), it was found that the disk follows Kepler’s laws, i.e. $v \propto r^{-1/2}$, as is to be expected if there were a point mass at the centre. From the velocities it was inferred that this invisible object has a mass of $3 \cdot 10^9 \, \text{M}_\odot$, again ruling out anything but a black hole.

Another example is NGC 4258, where heating of the gas by a central X-ray source causes some parts of the disk to act as an H₂O maser. The velocities of these gas blobs have been measured using radio telescopes and again follow a Keplerian profile corresponding to a central mass of $3.6 \cdot 10^7 \, \text{M}_\odot$.

Another, more generally applicable technique is to determine the velocity distribution of the stars by observing the stellar absorption lines in the spectra of galaxies. In this way, it is possible to perform a survey of a large group of nearby

---

¹Above a certain mass, estimated to be about $1.8$–$2.2 \, \text{M}_\odot$, depending on the equation of state, but certainly not higher than $3.2 \, \text{M}_\odot$ (see Rhoades and Ruffini (1974)), degeneracy pressure of the neutrons can no longer balance out the gravitational forces and the neutron star must collapse to a black hole.

²Two major uncertainties are the mass of the O-type companion star HDE 226868 and the value of the inclination.

³Some theories predict the possibility of other, more exotic objects, such as self-bound strange quark matter stars, but we will ignore these in this discussion.
galaxies. It turns out that every galaxy with a bulge contains a SMBH and that the ratio of the mass of the bulge and of the SMBH is a constant value. Even more interesting is the fact that there is a relationship between the mass of the SMBH and the velocity dispersion, the so-called $M$-$\sigma$ relation

$$M \propto \sigma^\alpha$$

where $\alpha = 4.24$ (see Gültekin et al. (2009)), as pictured in Fig. 3.1. Even stars outside the influence zone of the SMBH are still correlated to the mass of the black hole, which suggests that there is a close connection between the formation of the SMBH and the rest of the galaxy. However, there are still many uncertainties regarding the origin of SMBHs, see for example Adams et al. (2003).

### 3.3 Accretion

This section is based on Frank et al. (1985). Accretion is the increase of mass of a celestial body as material falls on to it. This process releases the gravitational potential energy of the infalling matter. According to Newton, the amount of energy is given by

$$\Delta E_{\text{acc}} = GMm/R^*$$

where $M$ and $R^*$ are the mass and radius of the stellar object and $m$ is the mass of the accreted matter. If all of this energy were converted into radiation, it would yield an accretion luminosity of

$$L_{\text{acc}} = GM\dot{M}/R^*$$

where $\dot{M}$ is the accretion rate. In the case of compact objects, i.e. for high ratios of $M/R^*$, accretion is a very efficient way of extracting energy from matter, for a neutron star even far more efficient than nuclear fusion. However, in the case of black holes, Eqn. 3.3 will not hold, as incoming matter will just fall through the Schwarzschild radius and be gone, adding to the mass of the black hole, but not producing any radiation. Introduce the efficiency $\eta$ as follows

$$L_{\text{acc}} = 2\eta GM\dot{M}/R^* = \eta\dot{M}c^2$$

where we have used the Schwarzschild radius for $R^*$.

Consider fully ionised hydrogen accreting on to an object, assuming a steady, spherically symmetric flow. As the matter accretes, it heats up and starts dissipating energy via radiation. The electrons and the protons form pairs. The
Figure 3.1: $M-\sigma$ relation. The symbols represent the mass measurement technique, the colours the type of galaxy. Figure taken from Gültekin et al. (2009).
radiation will exert a force on the free electrons through Compton scattering (the cross-section for the protons is negligible), equal to

$$F_{\text{rad}} = \frac{L}{4\pi r^2} \sigma_T c$$  \hspace{1cm} (3.5)

where $\sigma_T$ is the Thomson cross-section. Gravity pulls on the protons (this time the electrons can be neglected)

$$F_{\text{grav}} = \frac{GMm_p}{r^2}$$  \hspace{1cm} (3.6)

Equating the two forces ($F_{\text{rad}} = F_{\text{grav}}$) gives for the luminosity

$$L_{\text{Edd}} = \frac{4\pi GMm_p c}{\sigma_T}$$  \hspace{1cm} (3.7)

If the luminosity is greater than this value, called the Eddington luminosity, gravity can no longer overcome the radiation pressure and the accretion will stop.

In most binaries, mass transfer takes place at some time in their evolution. There are two ways mass from one star can be donated to the other: Roche lobe overflow and stellar wind accretion. Mass however cannot just flow from one star to the other, because it possesses angular momentum. Instead, an accretion disk is formed, which allows for the conversion of gravitational potential energy into radiation.

If we assume the matter is on a Keplerian orbit, the angular velocity is given by

$$\Omega_K(R) = \left(\frac{GM}{R^3}\right)^{1/2}$$  \hspace{1cm} (3.8)

so that a ring of mass $m$ at the surface of the compact object has a kinetic energy of

$$E = \frac{1}{2} \frac{GMm}{R_*}$$  \hspace{1cm} (3.9)

Comparing this to Eqn. 3.3 shows that the disk only converts half of the energy into radiation

$$L_{\text{disk}} = \frac{1}{2} L_{\text{acc}}$$  \hspace{1cm} (3.10)

In the case of a black hole, the other half is lost. In the next section, we will consider this problem using relativity and it will turn out that the situation is in fact more complex.
3.4 Motion of test particles and the stability of orbits

In this section we will study the motion of test particles and the stability of orbits, first for the Schwarzschild case and then for the Kerr black hole. The following is based on Shapiro and Teukolsky (1983). For simplicity we will be using natural units \( (G = c = 1) \) in this section.

### 3.4.1 Schwarzschild

Starting from Eqn. 2.42, we find the following Lagrangian

\[
2L = - \left( 1 - \frac{2M}{r} \right) \dot{t}^2 + \left( 1 - \frac{2M}{r} \right)^{-1} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2
\]

(3.11)

where a dot denotes differentiation with respect to the affine parameter\(^4\) \( \lambda = \tau/m \). For the canonically conjugate momenta we have

\[
p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha}
\]

(3.12)

Also

\[
g_{\alpha\beta} p^\alpha p^\beta = -m^2
\]

(3.13)

or in other words \( L = -m^2/2 \). If we choose our coordinates such that the trajectory of the particle stays in the plane, something that can be done without loss of generality because of the spherical symmetry, we can set \( \theta = \pi/2 \) to find the following equations of motion using Eqn. 3.12

\[
p_\phi = r^2 \dot{\phi} = l
\]

(3.14)

\[
-p_t = \left( 1 - \frac{2M}{r} \right) \dot{t} = E
\]

(3.15)

where the quantities \( l \) and \( E \) can be interpreted as respectively the angular momentum and the energy of the particle. Define their mass-normalised versions as follows

\[
\tilde{E} = \frac{E}{m}, \quad \tilde{l} = \frac{l}{m}
\]

(3.16)

\(^4\)An affine parameter is any parameter related to the proper time by the following transformation

\[
\tau \rightarrow \lambda = a\tau + b
\]

with \( a \) and \( b \) some constants.
We then get the following equations of motion

\[
\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - 2M/r} \tag{3.17}
\]

\[
\frac{d\phi}{d\tau} = \frac{\tilde{l}}{r^2} \tag{3.18}
\]

\[
\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2M}{r}\right) \left(1 + \frac{\tilde{l}^2}{r^2}\right) \tag{3.19}
\]

where the last equation comes from Eqn. 3.13. It can be rewritten as

\[
\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - V(r) \tag{3.20}
\]

when we introduce the effective potential

\[
V(r) = \left(1 - \frac{2M}{r}\right) \left(1 + \frac{\tilde{l}^2}{r^2}\right) \tag{3.21}
\]

Circular orbits must have

\[
\tilde{E}^2 = V(r), \quad \frac{\partial V}{\partial r} = 0 \tag{3.22}
\]

From this criterion, equations for \(E\) and \(l\) can be found

\[
\tilde{E} = \frac{(r - 2M)^2}{r(r - 3M)} \tag{3.23}
\]

\[
\tilde{l} = \frac{Mr^2}{r - 3M} \tag{3.24}
\]

Stability of the orbit requires that

\[
\frac{\partial^2 V}{\partial r^2} > 0 \tag{3.25}
\]

which gives \(r > 6M\), so that we have a limiting radius of

\[
r_{\text{ms}} = 6M \tag{3.26}
\]

This is the radius of marginal stability, also known as the innermost stable circular orbit (ISCO).
3.4.2 Kerr

We now repeat the analysis for rotating black holes. Assume that the particle stays in the equatorial plane \((\theta = \pi/2)\). The Lagrangian follows from Eqn. 2.63

\[
2L = - \left(1 - \frac{2M}{r}\right) \dot{t}^2 - \frac{4aM}{r} \dot{t} \dot{\phi} + \frac{r^2}{\Delta} \dot{r}^2 + \left(\frac{r^2 + a^2 + \frac{2Ma^2}{r}}{\Delta}\right) \dot{\phi}^2
\]  

Calculating the canonically conjugate momenta like before using Eqn. 3.12 and solving for \(\dot{t}\) and \(\dot{\phi}\) we find

\[
\dot{t} = \frac{(r^3 + a^2r + 2Ma^2)E - 2aMl}{r\Delta} \tag{3.28}
\]

\[
\dot{\phi} = \frac{(r - 2M)l + 2aME}{r\Delta} \tag{3.29}
\]

The third equation of motion is given by Eqn. 3.13. After plugging in the last two equations this becomes

\[
r^3 \left(\frac{dr}{d\lambda}\right)^2 = R(E, l, r) \tag{3.30}
\]

where

\[
R = E^2(r^3 + a^2r + 2Ma^2) - 4aMEl - (r - 2M)l^2 - m^2r\Delta \tag{3.31}
\]

Circular orbits must have

\[
R = 0, \quad \frac{\partial R}{\partial r} = 0 \tag{3.32}
\]

From this criterion, equations for \(E\) and \(l\) can be found

\[
\tilde{E} = \frac{r^2 - 2Mr + a\sqrt{Mr}}{r(r^2 - 3Mr + 2a\sqrt{Mr})^{1/2}} \tag{3.33}
\]

\[
\tilde{l} = \frac{\sqrt{Mr}(r^2 - 2a\sqrt{Mr} + a^2)}{r(r^2 - 3Mr + 2a\sqrt{Mr})} \tag{3.34}
\]

where the tilde indicates the mass-normalised version. Stability of the orbit requires that

\[
\frac{\partial^2 R}{\partial r^2} \leq 0 \tag{3.35}
\]

which gives

\[
1 - \tilde{E}^2 \geq \frac{2}{3} \frac{M}{r} \tag{3.36}
\]
With Eqn. 3.33 we get a quartic equation in \( r^{1/2} \). There is an analytical solution by Bardeen et al. (1972), from which we find that for a maximally spinning black hole \((a = 1)\), the radius of marginal stability or the ISCO is given by

\[
    r_{\text{ms}} = M \tag{3.37}
\]

Unfortunately, the analytical solution breaks down for \( a > 1 \). Numerical solutions, however, still exist and can be found using a simple Mathematica code:

```mathematica
Clear[r];
Clear[a];
M = 1;
radiuslist = List[];
energylist = List[];
f[till_] := Module[{a = 1},
    energy = (r^2 - 2 M r + a Sqrt[M r])/
               (r (r^2 - 3 M r + 2 a Sqrt[M r])^(1/2));
    radius = FindRoot[1 - energy^2 == (2 M)/(3 r), {r, 0.84}];
    AppendTo[radiuslist, {a, r /. radius}];
    AppendTo[energylist, {a, energy /. radius}];
    a = a + 0.001;
    If[a > till, Goto[end]];
    Goto[begin];
    Label[end];
]

f[2];
ListPlot[radiuslist, PlotRange -> {0, 2}]
ListPlot[energylist, PlotRange -> {-1, 1}]
```

The results are plotted in Fig. 3.2. As can be seen in these graphs, the ISCO continues to go down as we increase the spin of the black hole, until we reach the minimum radius of \( r = 2M/3 \) at a spin of

\[
    a/M = \sqrt{32/27} \approx 1.0886 \tag{3.38}
\]

after which the ISCO goes up again. Note that this minimum occurs at an over-rotation of only 9\%. At this spin, particles in the ISCO have an energy of zero, which means that they have lost all their energy, i.e. the accretion efficiency is 100\%.

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Figure 3.2: Left graph: radius of the ISCO as a function of the angular momentum. Right graph: energy of a particle at the ISCO as a function of the angular momentum.

**Momentum**

A geodesic has four constants of motion: the energy $E$, the angular momentum $L$, the rest mass $\mu$ and the Carter constant $Q$. Define $\lambda = L/E$ and $q^2 = Q/E^2$. According to Speith et al. (1995), we then have

\[
\begin{align*}
    p_t &= -E \\
    p_r &= \pm \frac{E}{\Delta} \sqrt{V_r} \\
    p_\theta &= \pm E \sqrt{V_\theta} \\
    p_\phi &= E\lambda
\end{align*}
\]  

where

\[
\begin{align*}
    V_r &= r^4 + (a^2 - \lambda^2 - q^2)r^2 + 2M \left((a - \lambda)^2 + q^2\right)r - a^2q^2 - \frac{\mu^2}{E^2} \Delta \\
    V_\theta &= q^2 - \cos^2(\theta) \left( \frac{\lambda^2}{\sin^2(\theta)} - a^2 + a^2\frac{\mu^2}{E^2} \right)
\end{align*}
\]  

These equations can be integrated with $\mu = 0$ to obtain the equations of motion for null geodesics. Because of the assumptions of a axial symmetry and stationary disk, we only need to consider the $r$ and $\theta$ coordinates, so we get

\[
\int_{r_e}^{\infty} \frac{dr}{\pm \sqrt{V_r}} = \int_{\pi/2}^{\infty} \frac{d\theta}{\pm \sqrt{V_\theta}}
\]  

where we integrate from $r_e$, the radius of emission, to infinity and from the equatorial plane, $\theta = \pi/2$, to the inclination angle $\theta_o$. 

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Accretion disk

According to Speith et al. (1995), the gas of the accretion disk has the four-velocity

\[ u_t = \frac{r\sqrt{r} + a\sqrt{M}}{\sqrt{r^2 - 3Mr + 2a\sqrt{M}\sqrt{r}}} \]

\[ u_r = 0 \]

\[ u_\theta = 0 \]

\[ u_\phi = \frac{\sqrt{M}}{r\sqrt{r} + a\sqrt{M}} \] \hspace{1cm} (3.42)

3.5 Iron line

In Reynolds and Nowak (2003) an explanation is given for the fluorescent iron line.

Relatively cold material in the near vicinity of an astrophysical black hole may be irradiated by a spectrum of hard X-rays. This can give rise to a spectrum of fluorescent emission lines, the most prominent one being the Kα line of iron, with an energy of 6.40–6.97 keV, depending upon the ionisation state of the iron.

Suppose we have a photon hitting a neutral atom. If the photon energy is high enough, this atom can be ionised by kicking out one of the electrons (photo-ejection). The electrons in the K-shell (the \( n = 1 \) shell) have the largest cross-sections for this process of photo-ionisation. We now have an ion, which has two ways of de-excitation. It starts with an electron from the L-shell (\( n = 2 \)) dropping into the K-shell. The excess energy can then either be radiated as a Kα line photon (fluorescence), or a L-shell electron can be ejected (autoionisation or the Auger effect). The fluorescent yield is the ratio of the probability that an excited ion will de-excite via fluorescence and the probability of autoionisation. It increases monotonically with the atomic number \( Z \).

The astrophysical model is as follows: suppose we look at an accretion disk consisting of gas, irradiated from above by a continuum X-ray spectrum produced in the disk corona via inverse Compton scattering. If the temperature is not too high, hydrogen and helium are fully ionised, while the metals (for an astronomer, this means all the other elements) are still neutral. Part of the impinging radiation will be reflected, which we will then observe in addition to the original spectrum. The results of an observation taken with Chandra can be found in Fig. 3.3. The fluorescent emission lines, caused by photo-ionisation of metals, can be seen. The Kα iron line at 6.40 keV is the strongest one, as iron has both a large fluorescent yield and a high cosmic abundance. Also note the Compton reflection hump or the Compton shoulder peaking at \( \sim 30 \) keV. It is caused by the Compton
Figure 3.3: The observed spectrum of the HMXB GX301-2, taken with the Chandra HETG. Visible are the Fe Kα and Fe Kβ fluorescence lines, the Fe K edge, the Compton shoulder and a hot line. Figure taken from Torrejón et al. (2010).
downscattering of iron line photons on the lower-energy electrons in the stellar wind of the companion star.

3.6 Speith code

The code by Speith et al. (1995), which is based on an earlier paper by Cunningham (1975) and was later modified again by Brenneman and Reynolds (2006), calculates and produces a line profile with the relativistic effects (mainly redshifts and beaming) taken into account. In our case, the emission line will be the Kα iron line at 6.40 keV, but the code is more general, starting from a normalised delta-function.

3.6.1 Introduction of the transfer function

The observed specific flux $F_{\text{obs}}^{(\nu)}$ is defined as

$$F_{\text{obs}}^{(\nu)} = \int I_{\nu_o} \cos \theta d\Omega$$

(3.43)

where $I_{\nu_o}$ is the observed specific intensity of the radiation. $\theta$ is the angle between the direction of emission and the line of sight. As the distance between the observer and the emitting object goes to infinity, we can put $\cos(\theta) \approx 1$. The integration is performed over the solid angle $\Omega$.

According to Liouville’s theorem the observed and emitted specific intensities are related by

$$\frac{I_{\nu_o}}{\nu_o^3} = \frac{I_{\nu_e}}{\nu_e^3}$$

(3.44)

where $\nu_o$ and $\nu_e$ are the observed and emitted frequencies respectively. It is convenient to introduce a parameter $g$ for the ratio of the two

$$g = \frac{\nu_o}{\nu_e}$$

(3.45)

In terms of this redshift $g$, we get for the flux

$$F_{\text{obs}}^{(\nu)} = \int g^3 I_{\nu_e} d\Omega$$

(3.46)

The intensity $I_{\nu_e} = I_{\nu_e}(\nu_e, r_e, n_e)$ is a function that depends only on the emitted frequency $\nu_e$, the emission radius $r_e$ and the direction of the emitted radiation, described by the polar angle $n_e$. Introduce a new parameter $g^*$, the relative redshift, defined as

$$g^* = \frac{g - g_{\text{min}}}{g_{\text{max}} - g_{\text{min}}}, \quad 0 \leq g^* \leq 1$$

(3.47)
All of these redshifts are a function of the radius of emission $r_e$ and the angle between the disk and the observer $\theta_o$, the so-called inclination. We can rewrite the expression for the flux by switching from an integration over the solid angle $\Omega$ to an integration over the emission radius $r_e$ and the relative redshift $g^*$. This can be done by introducing a transfer function $f$, defined as

$$f(g^*, r_e, \theta_o) = \frac{r_o^2}{\pi r_e^2} g \sqrt{g^*(1 - g^*)} \left| \frac{\partial \Omega}{\partial (g^*, r_e)} \right|$$  \hspace{1cm} (3.48)$$

(note our notation for the Jacobian). In terms of this transfer function, the specific flux reads

$$F^{(\nu)}_{\text{obs}} = \frac{1}{r_o^2} \int_{r_+}^\infty \frac{\pi r_e^2 g^2}{\sqrt{g^*(1 - g^*)}} f(g^*, r_e, \theta_o) I_{\nu_e}(\nu_e, r_e, n_e) dg^* dr_e$$  \hspace{1cm} (3.49)$$

The problem is now reduced to the calculation of the transfer function.

### 3.6.2 Intensity

The intensity $I$ is a function of $\nu_e$, $r_e$ and $n_e$, or alternatively

$$I = I(\nu_o/g, r_e, n_e)$$  \hspace{1cm} (3.50)$$

Now assume that the intensity can be expressed as follows

$$I = \delta(\nu_o/g - \nu_e) X(r_e, n_e)$$  \hspace{1cm} (3.51)$$

The Dirac delta function eliminates the integration over the redshift. We have

$$dg^* = \frac{1}{g_{\text{max}} - g_{\text{min}}} dg$$  \hspace{1cm} (3.52)$$

and in turn

$$dg = \frac{\nu_o}{\nu_e^2} d\nu_e$$  \hspace{1cm} (3.53)$$

so we end up with

$$F^{(\nu)}_{\text{obs}} = \frac{1}{r_o^2} \int_{r_+}^\infty \frac{\pi r_e^2 g^2}{\sqrt{g^*(1 - g^*)}} f(g^*, r_e, \theta_o) I_{\nu_e}(\nu_e, r_e, n_e) \frac{1}{g_{\text{max}} - g_{\text{min}}} \frac{\nu_o}{\nu_e^2} dr_e$$

$$= \frac{1}{r_o^2} \int_{r_+}^\infty \frac{\pi r_e^2 g^3}{\nu_e \sqrt{g^*(1 - g^*)}} g_{\text{max}} - g_{\text{min}} f(g^*, r_e, \theta_o) X(r_e, n_e) dr_e$$  \hspace{1cm} (3.54)$$

We now only have the integration over the radius left.
3.6.3 Emissivity

We still have to make an assumption about the emissivity. We choose a powerlaw, following Laor (1991)

\[ X(r_e, n_e) \propto Y(n_e) r_e^{-\beta} \]  

(3.55)

In this parametrisation, the limb-darkening is taken into account by \( Y(n_e) \), which does not depend on \( \nu_e \) or \( r_e \), and is arbitrarily modelled as

\[ Y(n_e) \propto 1 + 2.06 \cos n_e \]  

(3.56)

The value of \( \beta \) determines which part of the disk radiates the most: if \( \beta \) is large, the emission is dominated by the innermost parts of the disk, if \( \beta \) is smaller, a significant part of the radiation comes from the outer regions of the disk. In Section 3.6.7 we will see how the emissivity influences the line profile.

3.6.4 Calculation of the transfer function

Up to now, all we have been doing is rewriting the same problem in a different, and hopefully more convenient, notation. Now, let us focus on the actual calculation of the transfer function. This function depends only on \( r_e \) and \( g^* \) (of course, it also depends on \( a \) and \( \theta_o \), but these are fixed initial parameters).

Rewrite the Jacobian as

\[ r_o^2 \left| \frac{\partial \Omega}{\partial (g^*, r_e)} \right| = \frac{q(q_{\text{max}} - q_{\text{min}})}{\sin \theta_o \beta} \frac{\partial (\lambda, q)}{\partial (g, r_e)} \]  

(3.57)

Numerically, we can calculate the Jacobian by using finite differences

\[ \left| \frac{\partial (\lambda, q)}{\partial (g, r_e)} \right| = \left| \frac{\partial \lambda}{\partial g} \frac{\partial q}{\partial r_e} - \frac{\partial q}{\partial g} \frac{\partial \lambda}{\partial r_e} \right| \]  

(3.58)

Now all that is left to do is calculate \( \lambda \) and \( q \).

3.6.5 Calculation of \( \lambda \), \( q \) and \( g_{\text{min}} \) and \( g_{\text{max}} \)

The expression for the redshift \( g \) can be rewritten as a function of the four-momentum of the null geodesic at the emission point (see Eqn. 3.39) and the four-velocity of the emitting gas (see Eqn. 3.42)

\[ g = \frac{\nu_o}{\nu_e} = \frac{E_o}{E_e} = -\frac{E}{p_e u} \]  

(3.59)

Rewriting to get an expression for \( \lambda \) and simplifying, we get

\[ \lambda(g, r_e) = a + \sqrt{r_e} \left( r_e - \frac{1}{g} \sqrt{r_e(r_e - 3) + 2a \sqrt{r_e}} \right) \]  

(3.60)
In the code, this is calculated by the function \( \text{ginvrs} \).

Knowing \( \lambda \), Eqn. 3.41 becomes an implicit equation for \( q = q(g, r_e) \). This equation has to be solved numerically. If \( g_{\text{min}} < g < g_{\text{max}} \), there are two solutions, if \( g = g_{\text{min}} \) or \( g = g_{\text{max}} \) there is one solution and in all other cases no solution exists. Thus we can find the minimum and maximum values of the redshift by varying \( g \). We now also have a function \( q(g, r_e) \), and with the expression for \( \lambda \) we now have all the elements necessary to calculate the transfer function.

### 3.6.6 The code itself

The transfer function turns out to be a slowly varying function, as can be seen in Fig. 3.4. This allows us to calculate the transfer function for only a limited number of redshifts and then interpolate between them. This decreases computation time, while not really affecting the accuracy of the calculation.

An overview of the time spent in each function of the code is given in Table 3.1. We can see that the \( \text{gaulegnr} \) function, even though it takes very little time to run, takes up more than 80% of the the total code execution time, due to the large number of calls to this function. It effectively performs an integration, following the book on numerical analysis by Press et al. (1992).
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Figure 3.5: Iron emission line profiles for an accretion disk around a black hole for different values of the disk inclination $\theta_o$, generated using the Speith code. The black hole spin parameter is fixed at $\alpha = 0.9$, the emissivity index of the disk is fixed at $\beta = 0.5$.

### 3.6.7 Line profiles

In this section, we will take a look at the emission line profiles generated by the code. All of them have two peaks, because of the Doppler shifts caused by the rotating disk. One part of the disk comes towards us, leading to a blue shift, while the other part of the disk moves away from us, giving rise to a red shift. The relativistic beaming effect will enhance the blue peak over the red one. In addition to the line-of-sight Doppler effect, there are also the general relativistic effects of the gravitational potential and the special relativistic transverse Doppler effect. These will only dominate if the inclination is close to zero (i.e. in the case of a face-on disk), but can still lead to a significant broadening of the line.

Now let us discuss what happens to the spectral line profile as we change the three most important parameters, viz. the inclination, the emissivity and the spin parameter. For low values of the inclination (meaning the disk is more face-on), the line does not show the two peaked structure, as there is no line-of-sight Doppler shift. As we increase the inclination, the two peaks appear. The larger the inclination, the bigger the gap between the peaks, see Fig. 3.5.

As we have seen in the section on the emissivity, increasing the $\beta$ parameter leads to the emission coming more from the inner regions of the accretion disk. Here the high gravitational pull of the black hole causes a red wing in the
Figure 3.6: Iron emission line profiles for an accretion disk around a black hole for different values of the disk emissivity index $\beta$, generated using the Speith code. The black hole spin parameter is fixed at $\alpha = 0.5$, the inclination of the disk is fixed at $\theta_0 = 40^\circ$.

Figure 3.7: Iron emission line profiles for an accretion disk around a black hole for different values of the black hole spin parameter $\alpha$, generated using the Speith code. The inclination of the disk is fixed at $\theta_0 = 40^\circ$, the emissivity index is fixed at $\beta = 3$. 
The spin of the black hole has a similar effect. As we have already seen, increasing the spin brings the ISCO down, so that radiation is created nearer to the black hole, again giving rise to a red wing, extending to low energies, see Fig. 3.7.

3.7 Code extension

We will now give a description of the efforts taken to extend the code for usage with superspinning black holes, i.e. black holes with a spin parameter greater than one.

It was not possible to successfully extend the code, as all of the algorithms used in the original code were designed to work only for values of the spin parameter between 0 and 1. If the functions are given values outside of these bounds, they give bogus results. On top of that, the code is very long, written in Fortran, not very well designed, with insufficiently detailed documentation and comments (particularly on the numerical algorithms used) and badly written (e.g. long lists of if-statements). All of this makes it hard to modify the code for our purposes.

I have tried to follow the code as it is executed and to fix problems as they come up, modifying algorithms to work with a larger parameter space and introducing new cases. However, the problems I encountered were rarely local, but instead were “spread out” over multiple functions and would require the existing algorithms to be replaced with new ones.

With hindsight, it would probably have been more successful to do a complete rewrite of the code instead of trying to fix the already existing one. On the other hand, this would have taken longer than the time allotted.

3.8 Parallelisation

3.8.1 Introduction

OpenMP allows one to easily modify and parallelise an exciting piece of code, in our case the Speith code. Unlike Message Passing Interface (MPI) and Parallel Virtual Machine (PVM), OpenMP only works in the case of a shared memory, which boils down to one computer with multiple cores/processors. It does have the advantage though that it is far easier to configure compared to the other two, as the user does not have to setup anything. And with the advent of PC’s with eight cores, decent speedups can still be achieved, even on one machine.
It works as follows. There is one main thread that goes through the code. When it encounters the start of an OpenMP block, the main thread splits into several new threads, typically equal to the number of cores. The work in that block is then divided between those threads. At the end of the block, all threads come together again and the main thread continues.

There are several ways to divide the work between the threads. For-loops are ideally suited for this and using OpenMP this is a fairly easy task.

In order for a parallelisation to work, all functions need to be thread safe, meaning that the result of a call to a function does not depend on any previous calls. In our case, the cache functionality of the `gauleg` function had to be rewritten with this in mind.

As an example, here is some code:

```Fortran
!$OMP PARALLEL DEFAULT(PRIVATE), SHARED(RE,GMIN,GMAX)
!$OMP DO
  do i=1,nradii
    call gextrm(re(i),gmin(i),gmax(i))
    write (*,*) re(i),gmin(i),gmax(i)
  enddo
!$OMP END DO
!$OMP END PARALLEL
```

The first and last line indicate the block, here the main thread is split. Each of the new threads that are created takes a value of `i` and executes the code. When done, the thread takes a new value of `i` and executes the code again. When all the `i`’s have been used, the threads merge again. Note the first line, that is where we specify which variables are private and which are shared. In this case, all variables are private, except the arrays, as each thread only uses the `i`th element.

For more information about OpenMP, see Barney (2010).

### 3.8.2 Results

We run the code on a machine called *rabble*, with two 2.4 GHz dual-core AMD Opteron™ Processor 2216, i.e. four cores in total. Using the Linux `time` command, we can measure the speedup:

```
real    0m6.839s
user    0m19.969s
sys     0m0.028s
```

Dividing the user time by the real time gives us a speedup factor of about three. It is not four, as there are still some parts of the code that are run sequentially. This theoretical limitation is known as Amdahl’s law. There is no “overhead”, the total time (*user*) is the same as for the unparallelised version (20 seconds).
3.9 Cygnus X-1

3.9.1 Introduction

The X-ray source Cygnus X-1 is a high-mass X-ray binary (HMXB), i.e. a galactic black hole candidate (GBHC) with a mass donating secondary in a binary system. The spectrum of these systems can usually be described using some form of a powerlaw

\[ F(E) \propto E^{-\Gamma+1} \]  

(3.61)

where \( \Gamma \) is the photon index. The emission coming from these systems can roughly be divided into two kinds: soft states and hard states, as is explained in Reynolds and Nowak (2003). The interesting thing is that systems can switch between these two states. The difference is the slope of the powerlaw: the emission in the hard state is concentrated in the more energetic part of the spectrum (above 10 keV) and have a photon index \( \Gamma < 2 \), while the emission in the soft state is dominated by lower energies (less than 3 keV) with a photon index \( \Gamma > 2 \). In the soft state, the luminosity is higher and shows less variability ("soft/radio-loud state"), while in the hard state, the systems are less luminous and show more variable lightcurves ("hard/radio-quiet state"). This is the case for most stellar mass black hole systems, with the exception of Cygnus X-1, which has a more stable lightcurve. The physical causes of the state transitions are not yet completely clear. This is one of the reasons why we want to study the iron fluorescence line: it may be used as a probe of the accretion flow properties near the event horizon of the black hole.

In this section we will use the data of Nowak et al. (2010). The authors use three different models to explain the observed X-ray spectrum of Cygnus X-1, we will describe the fitting of an additional model, using the same data.

3.9.2 X-ray data analysis

Going from raw X-ray data to a well defined spectrum is not a trivial process. An X-ray telescope measures counts/exposure time/channel, while we are interested in the intensity or the energy/time/frequency/area/solid angle. To convert between the two, we use the following formula

\[ C(h) = \int_0^\infty \sum_i R_i(h, E) A_i(E) S_i(E) dE \Delta t + B(h) \]  

(3.62)

Here, \( S(E) \) is the spectral energy distribution (SED or photons/time/area/energy) of the source. \( C(h) \) is the total number of counts in channel or bin \( h \) and \( B(h) \) the number of background detections in that bin. \( R(h, E) \) and \( A(E) \) are respectively...
the detector response matrix and effective area. $\Delta t$ is the exposure time. The $i$
sums over multiple observations.

The effective area or ARF file takes into account the properties of the telescope,
e.g. filters, mirrors and the positional variability of the detector efficiency.

The response matrix (RMF) file converts between the number of counts and
the number of photons. It can also be used to map the channel/bin to the energy.
Note that there are two independent ways of measuring the energy: the position
of the light on the CCD after it went through the grating and the measurement
by the CCD itself.

Model spectra are of the form counts/time/area/bin. Interestingly, Eqn. 3.62
cannot just be inverted to give an expression for the flux, as that would blow up
the errors. Instead, the models are convolved with the detector response. The
model and the data are then compared in detector space. When plotting data, it
can be tempting to unfold the spectrum, i.e. to plot the real physical flux instead
of detector quantities (counts). These can be misleading however, as it is harder
to discern detector features.

ISIS (Interactive Spectral Interpretation System) is a piece of software designed
to help analyse and interpret high resolution X-ray spectra. See the manual
by Houck (2010). In practice, this means fitting a model to the data.

The $\chi^2$ can be used as an indicator of the goodness of fit, i.e. how well the
datapoints coincide with the model. The ultimate goal is to obtain a model that
accurately describes the physics going on. This model is assumed to be the one
with the lowest $\chi^2$. Sometimes several regions exist in parameter space with
approximately the same $\chi^2$. In that case, physical reasoning has to be used to pick
out the correct one. ISIS helps us by providing automated fitting methods, that
search the parameter space for better fits. A common problem is getting stuck in
a local minimum: small changes in any parameter only make the fit worse, even
though there is better fit out there, somewhere else in the parameter space. Still,
these fitting methods do a pretty good job, considering we are dealing with a,
say, thirteendimensional space. Error bar searches can help us get out of a local
minimum.

If we want to change a model parameter to a completely different value, a
good way is to fix that parameter and slowly increase or decrease its value.

3.9.3 The observations

The paper by Nowak et al. (2010) describes four observations of Cygnus X-1.
The fourth observation of this system, the one we will be using, was made on
2008-04-19 using three telescopes, viz. Suzaku, RXTE and Chandra. At this time,
the system was in the spectrally hard state (see the introduction of this section).
Figure 3.8: A sketch of a binary system. We see the black hole, surrounded by a corona and emitting two jets. The companion star transfers matter through a disk to the black hole. An observer has to look through the interstellar medium to see the emission from the system.

The Suzaku telescope has three instruments: XIS (covering the $\approx 0.3$–$10$ keV band), HXD-PIN (covering the $\approx 10$–$70$ keV band) and HXD-GSO (covering the $\approx 60$–$600$ keV band). For XIS, only the spectral energy ranges $0.8$–$1.72$ keV, $1.88$–$2.19$ keV and $2.37$–$7.5$ keV were used, as the response of the regions outside this range has not been well calibrated. The two holes correspond to Si and Ir features, caused by the detectors and the mirrors of the telescope. In the Fe Kα region, Suzaku has a high resolution of $E/\Delta E \approx 50$. This makes Suzaku a good compromise between broad spectral coverage and good spectral resolution.

RXTE consists of two instruments: the PCA (covering the $3$–$22$ keV band) and HEXTE (covering the $18$–$200$ keV band). At an energy of $6$ keV, it has a rather low resolution of $E/\Delta E \approx 6$.

For Chandra the HETG was used, consisting of the HEG (covering the $\approx 0.7$–$8$ keV band) and the MEG (covering the $\approx 0.4$–$8$ keV band). Chandra has the best spectral resolution, but only has a limited spectral coverage.

Some of the regions overlap, i.e. that part of the spectrum is covered by two telescopes/instruments. Details on the observations and the data reduction/analysis can be found in Nowak et al. (2010). This includes the creation of detector response matrices (RMF) and effective area (ARF, ancillary response function) files. All of the data were grouped, i.e. several channels were combined into one bin, to ensure a sufficiently high signal-to-noise ratio.
3.9.4 The model

A picture of the geometry that we are considering here can be found in Fig. 3.8. Let us briefly identify the different model components we shall use with the corresponding parts of the system. The disk produces a thermal spectrum, which is then upscattered by relativistic electrons in the corona through inverse Compton scattering and becomes a powerlaw. The jets, emitting synchrotron and synchrotron self-Compton radiation, can also be modelled as powerlaw. As we have already discussed in Section 3.5, the disk can reflect this radiation, leading to emission lines. The stellar wind coming from the companion star can add ionised absorption lines to light passing through it, the interstellar medium can be modelled as a neutral component.

The ultimate goal is to measure the physical properties of the black hole. But before we can do that, we also have to properly describe all the “nasty” astrophysics going on. It is often simpler to use a model that is more phenomenological, but hopefully still captures most of the physics. For example, many high energy processes can create a powerlaw, which itself contains little information and thus carries no clear signature of its physical origins.

In the models we use both a narrow and a broad iron emission line. The narrow line is emitted by the stellar wind of the secondary through fluorescence as it is illuminated by the X-ray source, the radiation reflected by the disk causes the broad line. Previous instruments, like RXTE, were not really able to separate the two, but with the more modern instruments, like Chandra and Suzaku, it becomes possible to get an idea of how much of the line is the narrow component and how much comes from the broad component. In this context, the paper by Torrejón et al. (2010) is interesting. The authors did a survey of 41 X-ray binaries using Chandra to study the narrow iron line. It turns out that almost all HMXBs have a Fe Kα fluorescence line, while such emission is very rare for LMXBs. This suggest that the neutral column of LMXBs is mainly the ISM, while for HMXBs the local absorption (wind of the donor) can also be significant.

By taking multiple observations, we can determine which parts of the system change over time. The spin of the black hole should remain stable, but the inner radius of the disk could very well change from observation to observation. Both of these will effect the shape of the broad iron line. Taking multiple observations can help to separate the narrow line from the broad line. Another thing that should remain stable over time is the interstellar column depth.

In ISIS, we can write our spectral model as follows:

\[
\text{constant(Isis\_Active\_Dataset)}*\text{TBnew(1)}*\text{lines(1)}* \\
( \text{gauss(1)}+\text{diskbb(1)}+\text{kerrconv(1,reflionx(1))} + \\
+ \text{highecut(1)}*\text{powerlaw(1)} )
\]
Let us go over each of the components in more detail.

The powerlaw is given a high-energy exponential cutoff (highecut), so it takes on the following form

$$F(E) \propto E^{-\Gamma+1} \exp\left(-E/E_c\right)$$

where $\Gamma$ is the photon index and $E_c$ is the cutoff energy.

TBnew by Wilms et al. (2000) models the neutral, interstellar absorption. The interstellar medium (ISM), consisting of gas, molecules and grains, absorbs part of the X-ray radiation. At every energy, the total photo-ionisation cross section of the gas component can be obtained by summing over all the individual cross sections of the elements and weighing their contributions by their abundances. For the molecular component, it suffices to only consider molecular hydrogen, as it has by far the highest abundance. The depletion of elements into grains lowers the total absorption, as grains are less effective in stopping radiation. The hydrogen number density $N_H$ is measured in atoms/cm$^2$. This value can also be obtained independently by using the 21 cm surveys of the galaxy, which measure the neutral column depth.

lines is the ionised absorption component. This is where the Chandra data come in, as the spectral resolution of Suzaku is not good enough to fit the absorption lines. The high spectral resolution of Chandra makes it possible to fit about 50–60 of these lines. Their relative strength was then fixed, so that in the fitting there is only one parameter, viz. the global normalisation factor.

gauss models a narrow Gaussian line at 6.40 keV, the energy of the iron fluorescence line, coming from stellar wind of the companion star.

diskbb models a so-called disk black body, see Mitsuda et al. (1984) and Makishima et al. (1986). It describes emission from an optically thick, geometrically thin, accretion disk, observed at an inclination angle $\theta_o$. The accretion disk consist of multiple rings with a certain temperature. Each of these rings emits its own black body radiation. Assume a Shakura and Sunyaev (1973) accretion disk. The temperature at a radius $r$ is then given by

$$T(r) = \left(3GM\dot{M}/8\pi\sigma r^3\right)^{1/4} \cdot \left(1 - (r_c/r)^{1/2}\right)^{1/4}$$

where $\sigma$ is the Stefan-Boltzmann constant and $r_c$ is a critical radius, within which the inward transfer of angular momentum takes place. Now we can integrate over all the different rings and obtain the composite photon spectrum:

$$f(E) = \frac{8\pi}{3} r_{in}^2 \cos i \int_{T_{in}}^{T_{out}} \left(T/T_{in}\right)^{-11/3} B(E, T) dT/T_{in}$$

58
Figure 3.9: A disk black body spectrum model, under the assumption that $T_{in} \gg T_{out}$. It can be seen that for $E \gtrsim 2k_B T_{in}$ the spectrum can be described as a black body, with a temperature of $0.71 T_{in}$ (dotted curve). For $E \lesssim 0.3k_B T_{in}$, the spectrum approximates a power law, with a photon index of $-2/3$ (dashed line). Figure taken from Makishima et al. (1986).

Here $B(E, T)$ is the blackbody photon flux per unit photon energy from a unit surface area of temperature $T$. The disk black body spectrum has the following properties. As $T_{out}$ is usually outside the energy range of interest, the spectrum is only determined by $r_{in} \cos^{1/2} i$ and $T_{in}$. For energies greater than $2k_B T_{in}$, with $k_B$ the Boltzmann constant, the spectrum can be described as a black body with a temperature of $0.71 T_{in}$. For $E \lesssim 0.3k_B T_{in}$ (and $E \gg k_B T_{out}$), the spectrum becomes a power law with a photon index of $-2/3$, or equivalently an energy index of $1/3$. The total luminosity can be found by integrating over the disk

$$L = \int_{r_{in}}^{r_{out}} 4\pi r^2 \sigma T^4(r) dr \approx 4\pi r_{in}^2 \sigma T_{in}^4,$$

assuming that $r_{out} \gg r_{in}$.

reflionx is a disk reflection model, see Ross and Fabian (2005). In Section 3.5 we have already introduced the general idea. The optically-thick atmosphere is assumed to have a constant density and to be in thermal and ionisation equilibrium. It is illuminated by radiation following a power-law spectrum that is given a high-energy exponential cutoff with an e-folding energy fixed at $E_c = 300$ keV and a sharp low-energy cutoff at 0.1 keV. The reflected spectrum is calculated between 1 eV–1 MeV. The model includes all abundant ionised species and their important transitions. The code has the following parameters.
**Fe/solar** (range 0.1 to 20.0): the abundance of iron in solar units. All the other elements are fixed at their solar values.

**Gamma** (range 1.4 to 3.3): the photon index for the impinging power-law spectrum.

**Xi** (range 10 to 10 000 erg cm/s): the ionisation parameter

\[
\xi = \frac{4\pi F}{n_H}
\]  

where \( F \) is the incoming flux and \( n_H \) the number density of hydrogen, fixed at \( 10^{-15} \text{cm}^{-3} \).

**Redshift**: the value of the redshift. In our case zero.

**Norm**: a normalisation constant.

The code does not work on-the-fly, but uses a look-up table with a precalculated grid of parameter values (in the case of reflionx, the table includes 1800 individual models). Let us go over the effects of changing the first three parameters.

Obviously, changing the ionisation parameter \( \xi \) has a profound influence on the reflected spectrum, see both Ross and Fabian (2005) and Reynolds and Nowak (2003). For values of \( \xi < 100 \) ergs cm/s, the gas is cold and neutral and the X-ray reflection contains a narrow iron fluorescence line at 6.4 keV. If we add more radiation, say \( 100 < \xi < 500 \) ergs cm/s, the iron is in the form of Fe xvi–Fe xii, with a vacancy in the L-shell (\( n = 2 \)). This means that the ions can easily absorb the K\( \alpha \) photons. Most of them will radiate away the excess energy by a fluorescent emission, effectively leading to a scattering. Eventually the scattering will end as the photon hits an ion that de-excites through the Auger effect. Because of this, only few fluorescent photons will exit the disk, leading to a very weak iron line. If we ionise the gas even further, so that \( 500 < \xi < 5000 \) ergs cm/s, the Auger effect cannot take place any more, because now there are less than the required 2 electrons in the L-shell. This means that, even though the photons are still subject to scattering, they can now leave the disk freely. This leads to K\( \alpha \) iron line emission of Fe xxv and Fe xxvi at an energy of 6.67 and 6.97 keV respectively. If we fully ionise the gas, i.e. \( \xi > 5000 \) ergs cm/s, the emission line completely disappears.

Besides changing the overall slope of the spectrum, the photon index \( \Gamma \) also has an effect on the ionisation, as harder spectra have more ionising power.

Increasing the abundance of iron has the expected behaviour: it enhances the Fe K\( \alpha \) fluorescence line.

The disk reflection model is convolved with kerrconv, which takes into account the effects of the rotation of the black hole. The kerrconv model is effectively
Figure 3.10: Top: the spectrum of the X-ray source Cygnus X-1, observed on 2008-04-19 using two telescopes, viz. Suzaku (XIS: 0.8–8 keV, HXD-PIN: 12–70 keV, HXD-GSO: 60–300 keV), RXTE (PCA: 3–22 keV, HXTE: 18–200 keV). The best-fit model is plotted as a red line. Bottom: the ratio between the fit and the datapoints.

an ISIS frontend to the Speith code. However, this version not only works on a single line, but on the entire reflected spectrum. The parameters are kept fixed as follows: emissivity index $\beta = 3$, black hole spin parameter $a = 0$, disk inclination $\theta_o = 30^\circ$, inner radius of the disk $r = 6r_{\text{ms}}$ and outer radius of the disk $r = 400r_{\text{ms}}$.

3.9.5 Results of the fitting

The data are plotted in Fig. 3.10. The best-fit parameters can be found in Table 3.2.
Table 3.2: Best-fit parameters of the dataset pictured in Fig. 3.10.

<table>
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3.9.6 Multiple Suzaku XIS observations of Cygnus X-1

This is another set of observations of the same system, Cygnus X-1, taken at different times around April 2009 using the Suzaku XIS. As we have already discussed above, taken multiple observations at different times allows us to get more information on which parts of the system change over time. For all of these datasets, we use a simpler model compared to the first one:

\[ \text{TBN} \times (\text{diskbb+powerlaw+gaussian+diskline}) \]

Here, \textit{diskline} is an ISIS frontend to the Speith code. It has four free parameters, viz. the normalisation, the line energy, the emissivity index \( \beta \) and the inner radius of the disk.

The data are plotted in Fig. 3.11–3.18. The best-fit parameters can be found in Tables. 3.3–3.10.
Figure 3.11: Dataset 1. Top: the spectrum of the X-ray source Cygnus X-1, observed around April 2009 using Suzaku XIS. Data are the blue circles, the best-fit model is plotted as a red line. Bottom: the ratio between the fit and the datapoints.

Table 3.3: Best-fit parameters of the dataset pictured in Fig. 3.11.

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Figure 3.12: Dataset 2. Top: the spectrum of the X-ray source Cygnus X-1, observed around April 2009 using Suzaku XIS. Data are the blue circles, the best-fit model is plotted as a red line. Bottom: the ratio between the fit and the datapoints.

Table 3.4: Best-fit parameters of the dataset pictured in Fig. 3.12.

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Figure 3.13: Dataset 3. Top: the spectrum of the X-ray source Cygnus X-1, observed around April 2009 using Suzaku XIS. Data are the blue circles, the best-fit model is plotted as a red line. Bottom: the ratio between the fit and the datapoints.

Table 3.5: Best-fit parameters of the dataset pictured in Fig. 3.13.

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Figure 3.14: Dataset 4. Top: the spectrum of the X-ray source Cygnus X-1, observed around April 2009 using Suzaku XIS. Data are the blue circles, the best-fit model is plotted as a red line. Bottom: the ratio between the fit and the datapoints.

Table 3.6: Best-fit parameters of the dataset pictured in Fig. 3.14.

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Figure 3.15: Dataset 6. Top: the spectrum of the X-ray source Cygnus X-1, observed around April 2009 using Suzaku XIS. Data are the blue circles, the best-fit model is plotted as a red line. Bottom: the ratio between the fit and the datapoints.

Table 3.7: Best-fit parameters of the dataset pictured in Fig. 3.15.

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Figure 3.16: Dataset 10. Top: the spectrum of the X-ray source Cygnus X-1, observed around April 2009 using Suzaku XIS. Data are the blue circles, the best-fit model is plotted as a red line. Bottom: the ratio between the fit and the datapoints.

Table 3.8: Best-fit parameters of the dataset pictured in Fig. 3.16.

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Figure 3.17: Dataset 13. Top: the spectrum of the X-ray source Cygnus X-1, observed around April 2009 using Suzaku XIS. Data are the blue circles, the best-fit model is plotted as a red line. Bottom: the ratio between the fit and the datapoints.

Table 3.9: Best-fit parameters of the dataset pictured in Fig. 3.17.

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Figure 3.18: Dataset 14. Top: the spectrum of the X-ray source Cygnus X-1, observed around April 2009 using Suzaku XIS. Data are the blue circles, the best-fit model is plotted as a red line. Bottom: the ratio between the fit and the datapoints.

Table 3.10: Best-fit parameters of the dataset pictured in Fig. 3.18.

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</table>
3.9.7 Discussion

In each of the fits, we find a value for the neutral column depth or the hydrogen number density \( N_H \) of about \( 0.6 \cdot 10^{22} \) atoms/cm\(^2\), which is not in disagreement with the 21 cm surveys of the galaxy. For the emissivity index \( \beta \) we find values of about \(-4\), which means that most of the emission is coming from the inner parts of the disk and indicates that the line has a red wing. This also agrees with the values we find for the inner radius of the disk. By looking at the normalisations of \texttt{diskline} and \texttt{gauss}, we can see that the relativistic disk line dominates the narrow Gaussian line.

All of these values are subject to a lot of systematic uncertainty, based upon the continuum model, but point in the direction that the narrow line alone is not enough to explain what is going on. We have to add a broad component, with a fair amount of the emission coming from the innermost regions of the disk. But before one can start constraining relativity, more work would have to be done to improve the continuum model and to get a better understanding of the individual components.

It would have been nice to have proper error bars on the values we found, but error bar searches take a very long time to run and are fraught with local minima.
Chapter 4

Conclusions

In this thesis, we have taken a look at superspinning black holes. These objects are interesting, because, if they exist, they would form a direct link between string theory and observations. We have introduced and discussed most of the important concepts of black holes in general relativity, such as black hole solutions and their properties, singularity theorems, cosmic censorship and black hole thermodynamics.

We have seen how the innermost stable circular orbit (ISCO) can be used as a natural inner truncation radius for the accretion disk. If we increase the spin, the ISCO goes down, thus increasing the amount of gravitational energy that can be freed through the process of accretion. For a Schwarzschild black hole \((a = 0)\), where \(a = J/M\) is the normalised spin parameter), the ISCO is located at 6 gravitational radii \((r = 6GM/c^2)\), yielding an efficiency of about 6%. For the maximally spinning Kerr solution \((a = 1)\), the ISCO is located at 1 gravitational radius \((r = GM/c^2)\), and the accretion efficiency goes up to about 42%. For superspinning black holes, the efficiency reaches 100% if we spin up the black hole only a little bit over the Kerr bound to \(a = \sqrt{32/27} \approx 1.0886\). At that value of the spin parameter, the ISCO goes down to its minimum value at \(r = 2GM/3c^2\).

We have looked into the physics of the iron fluorescence lines and seen how its profile changes as the parameters of the black hole and the accretion disk are modified. The line profile can show a two-peaked structure, as one part of the disk is moving towards us and the other part is moving away, causing the emitted radiation to experience Doppler shifts. Relativistic beaming can enhance the blue peak over the red peak. Emission coming from close to the black hole will experience gravitational redshifting. Thus, changing the emissivity index to make more radiation come from the inner regions of the disk will cause the line profile to show a red wing. Increasing the spin of the black hole has a similar effect: the ISCO and thus the inner radius of the disk goes down, more radiation...
comes from closer to the black hole, experiencing a gravitational redshift. I have tried to extend the code that calculates these line profiles to also work for superspinning black holes, but unfortunately I was not able to make it work. The code is old, written in Fortran, very long and its numerical algorithms are not well documented. It was designed and written to work only for values of the spin parameter between 0 and 1. With hindsight, it would probably have been better to do a clean rewrite of the code, instead of trying to make modifications. However, there are no theoretical reasons why the code should fail to run, it is purely a practical matter. I did manage to parallelise the code, getting a speedup factor of about three on a four-core machine.

Finally, we have studied the actual observational side to the story by looking at the X-ray binary Cygnus X-1. This system has been observed using three telescopes, viz. Suzaku, RXTE and Chandra. We have discussed the different parts of the system and the emission components that they emit. Summarised, the story is as follows: this disk creates a thermal spectrum, which is then upscattered by electrons in the corona through the process of inverse Compton scattering. This radiation then hits the disk, where neutral iron start emitting a broad fluorescence line. In addition to this broad line, the stellar wind can also create a narrow emission line through the same mechanism. All the emission then has to travel through the stellar wind of the companion and the interstellar medium, which also leaves an imprint on the observed spectrum. In order to accurately measure the shape of the broad iron emission line profile, one has to take into account and properly describe all the different components. Depending on the model that one uses, the profile of the iron line can change significantly. Still, we can draw some conclusions. Whatever model we choose, the data definitely tells us that the broad iron line is there. It also shows broadening and contains a significant red wing, implying that the emission comes from close to the black hole. This implies that the black hole in Cygnus X-1 is spinning, but currently, the data do no not allow a precise measurement.

In order to be able to make more accurate measurements, better data is required, as is usually the case in astronomy. Already the next generation of X-ray space telescopes are being designed. One example is the IXO (International X-ray Observatory), a joint effort of NASA, ESA and JAXA, see IXO (2010). This satellite would have far superior X-ray imaging, timing and spectroscopy capabilities than any of the current generation telescopes. Another space telescope for which there are plans is GRAVITAS (General Relativistic Astrophysics VIA Timing And Spectroscopy) by ESA, which is specifically being designed to study the energy, profile and variability of the Fe Kα fluorescence line, see GRAVITAS (2010). Should these two telescopes actually be built, far better observations could be carried out, so that the emission lines would be fully resolved, and our knowledge of black holes would undoubtedly increase tremendously. However, it is still
uncertain whether these satellites, which are both still in the concept phase, will ever be built, despite the obvious scientific importance of obtaining better spectra in the iron line region. And even if these plans are put through, it will still be decades before they come online.
Appendix A

Conformal diagrams

Conformal diagrams form a nice way of mapping complex spacetimes to a relatively simple diagram.

First of all, we need the concept of conformal transformations. A conformal transformation is a local change of scale. Since distances are measured by the metric, we can write this as

\[ \tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu} \quad \text{or} \quad d\tilde{s}^2 = \omega^2(x) ds^2 \quad (A.1) \]

where \( \omega(x) \) is some non-vanishing function and the tilde denotes the new metric. It turns out that null curves are left invariant by conformal transformations. This means that if some \( x^\mu(\lambda) \) is null in \( g_{\mu\nu} \), it will also be null in \( g_{\mu\nu} \). Proof: a curve is null if and only if its tangent vector \( dx^\mu/d\lambda \) is null, i.e. if

\[ g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} = 0 \quad (A.2) \]

But then it is also null in the conformally-related metric

\[ \tilde{g}_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} = \omega^2(x) g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} = 0 \quad (A.3) \]

This feature of conformal transformations makes them ideal for our purpose. If in our original metric light cones make an angle of \( 45^\circ \), then so will they in our new metric. We also want infinity to be a finite coordinate value away (otherwise we would quickly run out of paper!).

Now, as an example, let us find a conformal transformation that fulfils our needs for the Minkowski metric, which in polar coordinates reads

\[ ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \quad (A.4) \]

where

\[ d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \quad (A.5) \]
is the metric on a unit two-sphere.

First, switch to null coordinates

\[ u = t - r \]
\[ v = t + r \]  (A.6)

Then use the arctangent to bring infinity into a finite coordinate value

\[ U = \arctan u \]
\[ V = \arctan v \]  (A.7)

Then transform back to a timelike coordinate \( T \) and a radial coordinate \( R \)

\[ T = V + U \]
\[ R = V - U \]  (A.8)

with ranges

\[ 0 \leq R < \pi, \quad |T| + R < \pi \]  (A.9)

The new metric reads

\[ ds^2 = \omega^2(T, R)ds^2 = -dT^2 + dR^2 + \sin^2 Rd\Omega^2 \]  (A.10)

where

\[ \omega = \cos T + \cos R \]  (A.11)

We see that the old and the new metric are indeed conformally related. Since also
infinity is now a finite coordinate value away, the new metric can be drawn as a
conformal diagram, see Fig. A.1. Each point represents a two-sphere. Note that
indeed light cones are at ±45° throughout the diagram. The boundaries, referred
to as conformal infinity, can be divided into a couple of regions

\( i^+ = \) future timelike infinity \( (T = \pi, R = 0) \)
\( i^0 = \) spatial infinity \( (T = 0, R = \pi) \)
\( i^- = \) past timelike infinity \( (T = -\pi, R = 0) \)
\( J^+ = \) future null infinity \( (T = \pi - R, 0 < R < \pi) \)
\( J^- = \) past null infinity \( (T = -\pi + R, 0 < R < \pi) \)

Using conformal diagrams, we can quickly get an idea of the causal structure of
spacetime.
Figure A.1: The conformal diagram of Minkowski space. Figure taken from Carroll (2004).
Appendix B

Quantum field theory in curved spacetime

The Klein-Gordon Lagrangian in flat spacetime is

\[ \mathcal{L} = -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \]  

(B.1)

In curved spacetime this becomes

\[ \mathcal{L} = \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 - \xi R \phi^2 \right) \]  

(B.2)

where \( g^{\mu\nu} \) is the metric, \( g \) its determinant and \( R \) the curvature scalar, parametrised by a constant \( \xi \). There are two important choices for \( \xi \):

- minimal coupling: \( \xi = 0 \)
- conformal coupling: \( \xi = \frac{n-2}{4(n-1)} \) (\( = 1/6 \) in four dimensions) (B.3)

The conjugate momentum is then given by

\[ \pi = \frac{\partial \mathcal{L}}{\partial (\nabla_0 \phi)} = -\sqrt{g} \nabla_0 \phi \]  

(B.4)

We impose the canonical commutation relations

\[ [\phi(t, \mathbf{x}), \phi(t, \mathbf{x}')] = 0 \]
\[ [\pi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = 0 \]
\[ [\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = \frac{i}{\sqrt{-g}} \delta^{(n-1)}(\mathbf{x} - \mathbf{x}') \]  

(B.5)
The equation of motion for the scalar field is
\[ \Box \phi - m^2 \phi - \xi R \phi = 0 \]  
(B.6)

The inner product is defined as
\[ (\phi_1, \phi_2) = -i \int_\Sigma (\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1) n^\mu \sqrt{\gamma} d^{n-1} x \]  
(B.7)

We can find a set of solutions \( f_i(x^\mu) \) to Eqn. B.6 that are orthonormal
\[ (f_i, f_j) = \delta_{ij} \]  
(B.8)

and the conjugate modes with negative norm
\[ (f_i^*, f_j^*) = -\delta_{ij} \]  
(B.9)

Expand the fields as
\[ \phi = \sum_i (\hat{a}_i f_i + \hat{a}^\dagger_i f_i^*) \]  
(B.10)

where the coefficients \( \hat{a}_i \) and \( \hat{a}^\dagger_i \) have the following commutation relations
\[
\begin{align*}
[\hat{a}_i, \hat{a}_j] &= 0 \\
[\hat{a}^\dagger_i, \hat{a}^\dagger_j] &= 0 \\
[\hat{a}_i, \hat{a}^\dagger_j] &= \delta_{ij}
\end{align*}
\]  
(B.11)

There is a vacuum state that is annihilated by all annihilation operators
\[ \hat{a}_i |0_f\rangle = 0 \quad \forall i \]  
(B.12)

With this we define a entire Fock basis for the Hilbert space. A state with \( n_i \) excitations is obtained by acting \( n_i \) times with the creation operator \( \hat{a}^\dagger_i \) on the vacuum state
\[ |n_i\rangle = \frac{1}{\sqrt{n_i!}} (\hat{a}^\dagger_i)^{n_i} |0_f\rangle \]  
(B.13)

The number operator \( n_{f_i} = \hat{a}_i^\dagger \hat{a}_i \) counts the number of excitations of a mode.

Thus far, everything we have done seems familiar to the analysis for flat spacetime. The difference is the following: the modes of the basis, i.e. the \( f_i \), are not unique. We could just as well have chosen a different set of modes, say \( g_i \), with the same properties. They form a complete basis
\[ \phi = \sum_i (\hat{b}_i g_i + \hat{b}^\dagger_i g_i^*) \]  
(B.14)
where the creation and annihilation operators $\hat{b}^\dagger_i$ and $\hat{b}_i$ have the following commutation relations

\[
\begin{align*}
[\hat{b}_i, \hat{b}_j] &= 0 \\
[\hat{b}^\dagger_i, \hat{b}^\dagger_j] &= 0 \\
[\hat{b}_i, \hat{b}^\dagger_j] &= \delta_{ij}
\end{align*}
\] (B.15)

Again, there is a vacuum state that is annihilated by all annihilation operators

\[\hat{b}_i |0_g\rangle = 0 \quad \forall i \] (B.16)

Here too we can construct a Fock basis by letting creation operators act on the vacuum. We can define a number operator as well

\[n_{gi} = \hat{b}^\dagger_i \hat{b}_i \] (B.17)

**Bogolubov transformation**

The Bogolubov transformation is the expansion of one set of modes in terms of another

\[
\begin{align*}
g_i &= \sum_j (\alpha_{ij} f_j + \beta_{ij} f_j^*) \\
f_i &= \sum_j (\alpha_{ij}^* g_j - \beta_{ij} g_j^*)
\end{align*}
\] (B.18)

The matrices $\alpha_{ij}$ and $\beta_{ij}$ are the Bogolubov coefficients. By using the orthonormality of the mode functions it follows that

\[
\begin{align*}
\alpha_{ij} &= (g_i, f_j) \\
\beta_{ij} &= -(g_i, f_j^*)
\end{align*}
\] (B.19)

with normalisation conditions

\[
\begin{align*}
\sum_j (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) &= \delta_{ij} \\
\sum_j (\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}) &= 0
\end{align*}
\] (B.20)

There also exists a transformation between the operators

\[
\begin{align*}
\hat{a}_i &= \sum_j (\alpha_{ji} \hat{b}_j + \beta_{ji}^* \hat{b}_j^*) \\
\hat{b}_i &= \sum_j (\alpha_{ij}^* \hat{a}_j - \beta_{ij}^* \hat{a}_j^*)
\end{align*}
\] (B.21)
Now suppose that the system is in the $f$-vacuum $|0_f\rangle$. How many particles does an observer in the $g$-modes observe? Calculate the expectation value of the $g$ number operator in the $f$-vacuum.

\[
\langle 0_f | \hat{n}_g | 0_f \rangle = \langle 0_f | \hat{b}_i \hat{b}_i^\dagger | 0_f \rangle \\
= \langle 0_f | \sum_{jk} (\alpha_{ij} \hat{a}_j^\dagger - \beta_{ij} \hat{a}_j \hat{a}_k - \beta_{ik}^* \hat{a}_k^\dagger) | 0_f \rangle \\
= \sum_{jk} (-\beta_{ij})(-\beta_{ik}^*) \langle 0_f | \hat{a}_j \hat{a}_k^\dagger | 0_f \rangle \\
= \sum_{jk} \beta_{ij} \beta_{ik}^* \delta_{jk} \langle 0_f | (\hat{a}_j^\dagger \hat{a}_j + \delta_{jk}) | 0_f \rangle \\
= \sum_{j} \beta_{ij} \beta_{ij}^* \\
= \sum_{j} |\beta_{ij}|^2 \tag{B.22}
\]
Appendix C

Hawking radiation

There are several ways for deriving the Hawking temperature. A long one is by using the Unruh effect. A more efficient way is by using imaginary time.

C.1 Unruh effect

The following treatment is based on the book by Carroll (2004), Birrell and Davies (1982) and the lecture notes by Winitzki (2006). It is advisable to first take a look at Appendix B.

C.1.1 Rindler coordinates

For convenience, we do the calculation in two dimensions. It turns out that if we generalise to four dimensions, the results stay the same. The metric is given by

\[ ds^2 = -dt^2 + dx^2 \] (C.1)

Now, consider an observer, accelerating in the \( x \)-direction with a constant acceleration of magnitude \( \alpha \). The path \( x^\mu(\tau) \) followed is given by

\[ t(\tau) = \frac{1}{\alpha} \sinh(\alpha \tau) \]
\[ x(\tau) = \frac{1}{\alpha} \cosh(\alpha \tau) \] (C.2)

Proof:

\[ a^\mu = \frac{D^2 x^m u}{d\tau^2} = \frac{d^2 x^m u}{d\tau^2} \] (C.3)
as the Christoffel symbols vanish in flat spacetime. The components of $a^\mu$ thus are

$$
a^t = \alpha \sinh(\alpha \tau)
$$
$$
a^x = \alpha \cosh(\alpha \tau)
$$

(C.4)

with magnitude$^1$

$$
\sqrt{a_\mu a^\mu} = \sqrt{-\alpha^2 \sinh^2(\alpha \tau) + \alpha^2 \cosh^2(\alpha \tau)} = \alpha
$$

(C.5)

as we needed to show. The accelerating observer follows the following trajectory

$$
x^2(\tau) = t^2(\tau) + \frac{1}{\alpha^2}
$$

(C.6)

Thus, in the distant past and future this asymptotes to the null paths $x = -t$ and $x = t$.

Now, choose new coordinates $(\eta, \xi)$, known as Rindler coordinates, as follows

$$
t = \frac{1}{a} e^{a \xi} \sinh(a \eta)
$$
$$
x = \frac{1}{a} e^{a \xi} \cosh(a \eta)
$$

(C.7)

in the region of space where $x > |t|$. These new coordinates have ranges

$$
-\infty < \eta, \xi < +\infty
$$

(C.8)

The description of the path of Eqn. C.2 in the new coordinates can be found by equating the $t$ and $x$

$$
\eta(\tau) = \frac{\alpha}{a} \tau \\
\xi(\tau) = \frac{1}{a} \ln\left(\frac{\alpha}{a}\right)
$$

(C.9)

so $\eta$ is proportional to the proper time $\tau$ and the spatial coordinate $\xi$ is a constant. If $\alpha = a$, an observer moves as follows

$$
\eta = \tau, \quad \xi = 0
$$

(C.10)

The metric becomes

$$
ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)
$$

(C.11)

---

$^1$Remember the following identity: $\cosh^2 - \sinh^2 = 1$. 

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The part of Minkowski space where \( x > |t| \) is also known as Rindler space (region I). A Rindler observer is one moving along a constant-acceleration path.

The metric (Eqn. C.11) is independent of \( \eta \), which implies that \( \partial_\eta \) is a Killing vector. Expressing \( \partial_\eta \) in \((t,x)\) coordinates, we find

\[
\partial_\eta = \frac{\partial t}{\partial \eta} \partial_t + \frac{\partial x}{\partial \eta} \partial_x
= e^{a\xi} [\cosh(a\eta) \partial_t + \sinh(a\eta) \partial_x]
= a(x\partial_t + t\partial_x)
\]  

(C.12)

This is just the Killing field of a boost in the \( x \)-direction in Minkowski spacetime.

The redshift factor is defined as the norm of the Killing vector

\[
V = \sqrt{-\partial_\eta \partial^\eta} = \sqrt{-e^{2a\xi}(-\cosh(a\eta) \partial_t + \sinh(a\eta) \partial_x)} = e^{a\xi}
\]  

(C.13)

The surface gravity \( \kappa \) becomes

\[
\kappa = \sqrt{\nabla_\mu V \nabla^\mu V} = a
\]  

(C.14)

It is possible to extend the coordinates \((\eta, \xi)\) to the space where \( x < |t| \) (region IV) by flipping the signs in Eqn. C.7

\[
t = -\frac{1}{a} e^{a\xi} \sinh(a\eta)
\]

\[
x = -\frac{1}{a} e^{a\xi} \cosh(a\eta)
\]  

(C.15)

In Rindler coordinates, the massless Klein-Gordon equation takes the following form

\[
\Box \phi = e^{-2a\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi = 0
\]  

(C.16)

As we need our modes to be positive-frequency with respect to a future-directed Killing vector, we introduce two sets of modes

\[
g^{(1)}_k = \begin{cases}
\frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta + ik\xi} & \text{I} \\
0 & \text{IV}
\end{cases}
\]

\[
g^{(2)}_k = \begin{cases}
0 & \text{I} \\
\frac{1}{\sqrt{4\pi\omega}} e^{+i\omega\eta + ik\xi} & \text{IV}
\end{cases}
\]

with \( \omega = |k| \). Check that the modes are indeed positive-frequency w.r.t. future-directed Killing vector.

\[
\partial_\eta g^{(1)}_k = -i\omega g^{(1)}_k
\]

\[
\partial_{-\eta} g^{(2)}_k = -\partial_\eta g^{(2)}_k = -i\omega g^{(2)}_k
\]  

(C.17)
Expand the solution as follows

\[ \phi = \int dk \left( \hat{b}_k^{(1)} g_k^{(1)} + \hat{b}_k^{(1)*} g_k^{(1)*} + \hat{b}_k^{(2)} g_k^{(2)} + \hat{b}_k^{(2)*} g_k^{(2)*} \right) \]  

(C.18)

Let us check that the inner product (Eqn. B.7) is properly normalised. In the Rindler metric (Eqn. C.11), the future-directed unit normal to the surface \( \eta = 0 \) is normalised to

\[ -1 = g_{\mu\nu} n^\mu n^\nu = -e^{2a\xi} (n^0)^2 \]  

or

\[ n^0 = e^{-a\xi} \]  

(C.19)

The spatial metric determinant is

\[ \sqrt{\gamma} = e^{a\xi} \]  

(C.20)

so that conveniently \( n^0 \sqrt{\gamma} = 1 \) and

\[
\begin{align*}
(g_k^{(1)} : g_k^{(1)}) &= \delta(k_1 - k_2) \\
(g_k^{(2)} : g_k^{(2)}) &= \delta(k_1 - k_2) \\
(g_k^{(1)} : g_k^{(2)}) &= 0
\end{align*}
\]  

(C.22)

The Minkowski and Rindler vacuum states for are different. What appears to be empty for an observer in Minkowski space is full of particles for a Rindler observer, and vice versa. We could now calculate the Bogolubov coefficients, see Appendix B, but there is a faster way due to Unruh. From Eqn. C.7 and C.15

\[
e^{-a(\eta - \xi)} = \begin{cases} 
a(-t + x) & \text{I} \\
a(t - x) & \text{IV} 
\end{cases}
\]

\[
e^{a(\eta + \xi)} = \begin{cases} 
a(t + x) & \text{I} \\
a(-t - x) & \text{IV} 
\end{cases}
\]

We can now write is \( k > 0 \), i.e. if \( w = k \):

\[
\sqrt{4\pi \omega} g_k^{(1)} = e^{-i\omega \eta + ik\xi} = e^{-i\omega(\eta - xi)} = a^{i\omega/a} (-t + x)^{i\omega/a}
\]  

(C.23)

Do the same for \( g_k^{(2)} \):

\[
\sqrt{4\pi \omega} g_k^{(2)} = e^{+i\omega \eta + ik\xi} = e^{+i\omega(\eta + xi)} = a^{-i\omega/a} (-t - x)^{-i\omega/a}
\]  

(C.24)
which is not yet quite what we want. Take the complex conjugate and reverse the wave number.

\[
\sqrt{4\pi\omega} g^{(2)*}_{-k} = e^{-i\omega\eta + ik\xi} = e^{-i\omega(\eta-x)i} = a^{i\omega/a}(t-x)^{i\omega/a} = a^{i\omega/a}[e^{-i\pi(a^{-1})(-t+x)}]^{i\omega/a} = a^{i\omega/a} e^{i\pi/a}(-t+x)^{i\omega/a}
\]

(C.25)

The combination of the two

\[
\sqrt{4\pi\omega}(g^{(1)}_{k} + e^{-\pi\omega/a} g^{(2)*}_{-k}) = 2a^{i\omega/a}(-t+x)^{i\omega/a}
\]

(C.26)

is well defined on the whole space. A properly normalised version of this mode is given by

\[
h^{(1)}_{k} = \frac{1}{\sqrt{2 \sinh(\frac{\pi\omega}{a})}}(e^{\pi\omega/2a} g^{(1)}_{k} + e^{-\pi\omega/2a} g^{(2)*}_{-k})
\]

(C.27)

The corresponding extension for the \(g^{(2)}_{k}\) modes reads

\[
h^{(2)}_{k} = \frac{1}{\sqrt{2 \sinh(\frac{\pi\omega}{a})}}(e^{\pi\omega/2a} g^{(2)}_{k} + e^{-\pi\omega/2a} g^{(1)*}_{-k})
\]

(C.28)

The normalisation can be checked by using Eqn. C.22.

Now expand the field in these new modes

\[
\phi = \int dk \left( \hat{c}^{(1)}_{k} \hat{b}^{(1)}_{k} + \hat{c}^{(1)*}_{k} \hat{b}^{(1)*}_{k} + \hat{c}^{(2)}_{k} \hat{b}^{(2)}_{k} + \hat{c}^{(2)*}_{k} \hat{b}^{(2)*}_{k} \right)
\]

(C.29)

In our discussion of the Bogolubov transformations (see Appendix B) we have seen that the expressions of the \(h^{(1,2)}_{k}\) modes in terms of \(g^{(1,2)}_{k}\) implies corresponding expressions of the operators \(\hat{b}^{(1,2)}_{k}\) in terms of \(\hat{c}^{(1,2)}_{k}\), as follows

\[
\hat{b}^{(1)}_{k} = \frac{1}{\sqrt{2 \sinh(\frac{\pi\omega}{a})}}(e^{\pi\omega/2a} \hat{c}^{(1)}_{k} + e^{-\pi\omega/2a} \hat{c}^{(2)*}_{k})
\]

\[
\hat{b}^{(2)}_{k} = \frac{1}{\sqrt{2 \sinh(\frac{\pi\omega}{a})}}(e^{\pi\omega/2a} \hat{c}^{(2)}_{k} + e^{-\pi\omega/2a} \hat{c}^{(1)*}_{k})
\]

(C.30)

We are now curious to find out what an observer in Rindler space will see of the
Minkowski vacuum. Calculate the following

\[ \langle 0_M | a_R^{(1)}(k) | 0_M \rangle = \langle 0_M | b_k^{(1)} \hat{b}_k^{(1)} | 0_M \rangle \]

\[ = \frac{1}{2 \sinh(\frac{\pi \omega}{a})} \langle 0_M | e^{-\pi \omega / a} c_+^{(1)} c_-^{(1)} | 0_M \rangle \]

\[ = \frac{1}{2 \sinh(\frac{\pi \omega}{a})} \delta(0) \]

\[ = \frac{1}{e^{2\pi \omega / a} - 1} \delta(0) \]  
(C.31)

where we used the fact that \( c_k^{(1)} | 0_M \rangle \) is normalised

\[ \langle 0_M | c_k^{(1)} c_k^{(1)} | 0_M \rangle = \delta(0) \]  
(C.32)

We immediately recognise Eqn. C.31 as a Planck spectrum with temperature

\[ T = \frac{a}{2\pi} \]  
(C.33)

This is the Unruh effect:

An observer moving with a constant acceleration through empty Minkowski space will see a thermal spectrum.

### C.1.2 Hawking radiation

The redshift can be calculated as follows

\[ \omega_2 = \frac{V_1}{V_2} \omega_1 \]  
(C.34)

As \( V = e^{a\xi} \) we have \( \omega_2 = e^{a(\xi_1 - \xi_2)} \omega_1 \). The temperature then shifts as

\[ T = \frac{ae^{a(\xi_1 - \xi_2)}}{2\pi} \]  
(C.35)

Interestingly, we can apply the Unruh effect, as calculated in flat spacetime, to black holes. We have

\[ V = \sqrt{1 - \frac{2GM}{r}} \]  
(C.36)

and

\[ a = \frac{GM}{r\sqrt{r - 2GM}} \]  
(C.37)
Near the horizon \( r \approx 2GM \) the acceleration becomes large compared to the Schwarzschild radius: \( a \gg \frac{1}{2GM} \). We can consider the area near to the black hole to be flat. An observer at a fixed distance \( r_1 \) then sees Unruh radiation with a temperature of \( T_1 = a_1/2\pi \), as he is accelerating to avoid falling into the black hole. Further away from the horizon the assumption breaks down. No problem, as we can calculate what an observer at infinity will see by taking the redshift into account.

\[
T_2 = \frac{V_1}{V_2} T_1 = \frac{V_1 a_1}{V_2 2\pi} \quad \text{(C.38)}
\]

As \( r \to \infty \), \( V_2 \to 1 \), so the measured temperature becomes

\[
T = \frac{V_1 a_1}{2\pi} = \frac{\kappa}{2\pi} \quad \text{(C.39)}
\]

where \( \kappa \) is the surface gravity; for Schwarzschild this is \( \kappa = \frac{1}{4GM} \). Eqn. C.39 is the Hawking temperature.

C.2 Imaginary time

We will now redo the calculation using the imaginary time method.

C.2.1 Schwarzschild

The Schwarzschild metric is given by (see Eqn. 2.42)

\[
ds^2 = -(1 - \frac{R_S}{r}) dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad \text{(C.40)}
\]

We change the time coordinate to an imaginary one

\[
t \to i\tau \quad \text{(C.41)}
\]

which yields the so-called Euclidean metric

\[
ds^2 = \left(1 - \frac{R_S}{r}\right) d\tau^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad \text{(C.42)}
\]

This metric has a singularity at \( r = R_S \). This not a coordinate singularity, i.e. we cannot solve the problem by choosing different coordinates, as we did for Lorentzian metric. Now write the term in front of the \( dr^2 \) as \((dx/dr)^2\). We focus our attention to the horizon, as that is where the problem lies. Evaluate the expression around \( r = R_S \)

\[
\frac{dx^2}{dr^2} = \left(1 - \frac{R_S}{r}\right)^{-1} = \frac{r}{r - R_S} \approx \frac{R_S}{r - R_S} \quad \text{(C.43)}
\]
Integration yields

\[ x^2 = 4R_S(r - R_S) = 8GM(r - 2GM) = 16G^2M^2\left(1 - \frac{2GM}{r}\right) \]  
(C.44)

The metric now becomes

\[ ds^2 = \frac{x^2}{16G^2M^2}d\tau^2 + dx^2 + R_S^2d\Omega^2 \]  
(C.45)

Introducing a new variable \( d\tilde{\tau} = d\tau / 16G^2M^2 \) we get the metric of a circle

\[ ds^2 = x^2d\tilde{\tau}^2 + dx^2 + R_S^2d\Omega^2 \]  
(C.46)

that is, if we choose the \( \tilde{\tau} \) to be periodic. To eliminate the conical singularity we have to identify \( \tilde{\tau} \) with \( \tilde{\tau} + 2\pi \). This gives \( \beta = 4GM \cdot 2\pi \), so we find for the Hawking temperature

\[ T_H = \frac{1}{8\pi GM} \]  
(C.47)

This is by far the easiest way of deriving the Hawking temperature.

### C.2.2 Kerr-Newman

The following is based on an article by Mann and Solodukhin (1996). The Kerr-Newman metric is a combination of the Kerr and Reissner-Nordström metrics and can be written as follows (if we choose \( G = 1 \))

\[ ds^2 = g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 \]  
(C.48)

\[ g_{rr} = \frac{\rho^2}{\Delta}, g_{\theta\theta} = \rho^2, g_{tt} = -\frac{(\Delta - a^2\sin^2\theta)}{\rho^2} \]  
(C.49)

\[ g_{t\phi} = -\frac{a\sin^2\theta(r^2 + a^2 - \Delta)}{\rho^2}, g_{\phi\phi} = \left(\frac{(r^2 + a^2)^2 - \Delta a^2\sin^2\theta}{\rho^2}\right)\sin^2\theta \]  
(C.50)

\[ \Delta(r) = r^2 + a^2 + q^2 - 2mr, \quad \rho^2 = r^2 + a^2\cos^2\theta \]  
(C.51)

Again the event horizon can be found by setting \( g_{rr} \) equal to zero. We find

\[ r_\pm = m \pm \sqrt{m^2 - a^2 - q^2} \]  
(C.52)

So we can write \( \Delta(r) \) as

\[ \Delta = (r - r_+)(r - r_-) \]  
(C.53)
Again we make the transition to imaginary time

\[ t \rightarrow i\tau \]  

(C.54)

but now we also do a parameter transformation

\[ a \rightarrow i\hat{a}, \quad q \rightarrow i\hat{q} \]  

(C.55)

The Euclidean Killing vectors \( K \) and \( \tilde{K} \) and their corresponding one-forms \( \omega \) and \( \tilde{\omega} \) take the following form

\[ K = \partial_\tau - \frac{\hat{a}}{r^2 - \hat{a}^2} \partial_\phi, \quad \tilde{K} = \hat{a} \sin^2 \theta \partial_\tau + \partial_\phi \]  

(C.56)

\[ \omega = \frac{r^2 - \hat{a}^2}{\hat{\rho}^2} (d\tau - \hat{a} \sin^2 \theta d\phi), \quad \tilde{\omega} = \frac{r^2 - \hat{a}^2}{\hat{\rho}^2} (d\phi + \frac{\hat{a}}{r^2 - \hat{a}^2} d\tau) \]  

(C.57)

The Euclidean metric then becomes

\[ ds^2_E = \frac{\hat{\rho}^2}{\Delta} dr^2 + \frac{\hat{\Delta} \hat{\rho}^2}{(r^2 - \hat{a}^2)^2} + \hat{\rho}(d\theta^2 + \sin^2 \theta \tilde{\omega}^2) \]  

(C.58)

where we have used that

\[ \hat{\Delta} = r^2 - \hat{a}^2 - \hat{q}^2 - 2mr, \quad \hat{\rho}^2 = r^2 - \hat{a}^2 \cos^2 \theta \]  

(C.59)

The roots of \( \hat{\Delta} \) are given by

\[ \hat{r}_\pm = m \pm \sqrt{m^2 + \hat{a}^2 + \hat{q}^2} \]  

(C.60)

As before, we consider the metric around the horizon \( \Sigma \) at \( r = \hat{r}_+ \). Introducing a new radial variable \( x \) such that near the horizon we have

\[ \hat{\Delta} = \gamma (r - \hat{r}_+) = \frac{\gamma^2 x^2}{4} \]  

(C.61)

where

\[ \gamma = 2\sqrt{m^2 + \hat{a}^2 + \hat{q}^2} \]  

(C.62)

The metric then reads

\[ ds^2_E = ds^2_\Sigma + \hat{\rho}_+^2 \left( dx^2 + \frac{\gamma^2 x^2}{4(r_+^2 - \hat{a}^2)^2} \omega^2 \right) \]  

(C.63)

where \( \hat{\rho}_+^2 = r_+^2 - \hat{a}^2 \cos^2 \theta \) and where we have introduced

\[ ds^2_\Sigma = \hat{\rho}_+^2 (d\theta^2 + \sin^2 \theta \tilde{\omega}^2) \]

\[ = \hat{\rho}_+^2 d\theta^2 + \frac{(\hat{r}_+^2 - \hat{a}^2)^2}{\hat{\rho}_+^2} \sin^2 \theta d\psi^2 \]  

(C.64)
as the metric on the horizon surface $\Sigma$. The new angle coordinate $\psi$ is given by

$$\psi = \phi + \frac{\hat{a}}{(\hat{r}^2_+ - \hat{a}^2)} \tau$$  \hspace{1cm} (C.65)

We can now identify $\psi$ and $\psi + 2\pi$. The Euclidean metric (Eqn. C.63) can be further rewritten as

$$ds^2_E = ds^2_\Sigma + \hat{\rho}^2_+ ds^2_{C_2}$$  \hspace{1cm} (C.66)

where $ds^2_{C_2}$ is the metric of a two-dimensional disk $C_2$ attached to the horizon $\Sigma$ at a point $(\theta, \psi)$:

$$ds^2_{C_2} = dx^2 + \frac{\gamma^2 x^2}{4\hat{\rho}^4_+} (d\tau - \hat{a} \sin^2 \theta d\phi)^2.$$  \hspace{1cm} (C.67)

After the introduction of yet another angle coordinate, now on $C_2$,

$$\chi = \tau - \hat{a} \sin^2 \theta \phi$$  \hspace{1cm} (C.68)

the metric reads

$$ds^2_{C_2} = dx^2 + \frac{\gamma^2 x^2}{4\hat{\rho}^4_+} d\chi^2.$$  \hspace{1cm} (C.69)

To fix the conical singularity at $x = 0$ we have to identify the points $\chi$ and $\chi + 4\gamma^{-1} \hat{\rho}^2_+$. In order for this to hold independently of the coordinate $\theta$ on the horizon we must also identify points $(\tau, \phi)$ with $(\tau + 2\pi \beta_H, \phi - 2\pi \Omega \beta_H)$, where the angular velocity is given by

$$\Omega = \frac{\hat{a}}{(\hat{r}^2_+ - \hat{a}^2)}.$$  \hspace{1cm} (C.70)

and the Hawking temperature reads

$$\beta_H = \frac{(\hat{r}^2_+ - \hat{a}^2)}{\sqrt{m^2 + \hat{a}^2 + q^2}}.$$  \hspace{1cm} (C.71)

The identified points have the same coordinate $\psi$.

### C.2.3 Rindler

The following derivation can be found in Liu and Ma (1999). In Rindler coordinates, the time coordinate is given by (see Eqn. C.7)

$$t = \frac{1}{a} e^{\omega \xi} \sinh(a\eta)$$  \hspace{1cm} (C.72)
Now let
\[ \eta \rightarrow -i\eta \]  \hspace{1cm} (C.73)

Then time becomes imaginary
\[ t = \frac{1}{a\xi} e^{\alpha \xi} \sinh(i\alpha \eta) \]  \hspace{1cm} (C.74)

with a period of \(2\pi/a\), so the Unruh temperature becomes
\[ T_U = \beta^{-1} = \frac{a}{2\pi} \]  \hspace{1cm} (C.75)

This provides a nice shortcut to Eqn. C.33.
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Zwarte gaten, zoals beschreven door Einsteins algemene relativiteitstheorie, kunnen impulsmoment bezitten, maar slechts een beperkte hoeveelheid: te veel impulsmoment en de waarneemhorizon van het zwarte gat verdwijnt, hetgeen leidt tot een naakte singulariteit, d.w.z. een singulariteit die niet verborgen zit achter een waarneemhorizon. Het vermoeden van kosmische censuur stelt dat dit nooit zou moeten gebeuren. Sommige theorieën voorspellen echter dat er mogelijk supersymmetrische objecten bestaan die sneller draaien dan de relativistische bovengrens. We zullen kijken naar de fluorescentielijnen van ijzer, die zijn waargenomen in de spectra uitgezonden door zwarte gaten. De profielen van deze lijnen kunnen een vingerafdruk dragen van de relativistische effecten van het zwarte gat, hetgeen ons mogelijk in staat stelt de draaiing te bepalen. De lijnprofielen kunnen berekend worden met een computercode, die we vervolgens gebruiken om een model proberen te vinden voor het waargenomen spectrum van het röntgendubbelstersysteem Cygnus X-1.
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After my return, at the end of March 2010, the writing down of the astronomical research proved a bit harder than I had imagined. I should like to thank Sera Markoff for her patience.

Peter van Ham
Diemen, October 2010
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