Cosmic Strings

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August 13, 2009
To: C. Kelderman (1923 - 2008)

Thanks to: Dr. J.P. van der Schaar, O. Smits, I.S. Eliens, family and friends for their inspiration and support.
Summary

This thesis will give an introduction on the topic of cosmic strings. A cosmic string is a topological defect that arises during phase transitions. In the early universe, when the temperature drops below a certain critical temperature, phase transitions can occur, forcing the fields into a particular vacuum state. A cosmic string is a one-dimensional defect, but defects of other dimensions like monopoles and domain walls can also occur. In the context of the Abelian-Higgs model, a simple formulation of a cosmic string will be derived. There are several ways to detect cosmic strings. In this thesis observational effects due to gravitational lensing and gravitational waves will be discussed. Using General Relativity we will derive the Einstein equations in the presence of a cosmic string. In addition, the energy-momentum tensor will be derived from the effective string action. We will apply the Gauss-Bonnet theorem in order to calculate the wedge created in space-time caused by the presence of the string. These wedges act as gravitational lenses that can be observed. The second type of observational effects is caused by gravitational waves that can be emitted either by contracting loops, or cusps. Loops are formed when strings intersect with themselves or others in the network. String networks have some remarkable properties implying that the networks are scale invariant. Finally we describe some possible observations.
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1 Introduction

Cosmic strings are a specific type of topological defects that arise during phase transitions. Topological defects can occur when the field symmetries are broken. This means that the ground state does not exhibit the same symmetry as the full theory. Symmetry breaking happens when the universe cools down below some critical temperature $T_c$ and the field is forced to choose a vacuum state.

Due to symmetry breaking, energy can get trapped in specific regions of space. The topological structure of this trapped energy determines the nature of the defect. A line-like structure is called a cosmic string.

The universe, in its very early history, was an intense environment. During the initial stages of the development of the universe, phase transitions occurred that might have left traces that are still visible today. Phase transitions may have led to symmetry breaking and thus the formation of topological defects. Besides line-like defects, the cosmic string, other defects can occur, namely, domain walls or monopoles. Some of these defects may have survived to present day. With every broken symmetry there is a possibility for a topological defect, which could lead to the formation of cosmic strings.

As the universe cooled down it went through at least three phase-transitions:

1. The GUT transition occurs between $10^{-37}$s and $10^{-35}$s after the Big Bang. The Grand Unification Theory (GUT) predicts that at very high-energy scales the electroweak-nuclear and strong-nuclear forces are unified into one force. The GUT symmetries are broken by the rapid expansion that caused a cooling down of the universe.

2. Around $10^{-11}$s after the Big Bang the electroweak symmetry was broken. The electroweak symmetry unified electromagnetism and the weak interaction.

3. The quark-hadron transition at $10^{-5}$s after the Big Bang caused the plasma of free quarks and gluons to convert into hadrons (baryons and mesons, the more well-known baryons are protons and neutrons).

The Cosmological Principle states that on large scales the universe is homogeneous and isotropic: uniform in all directions. But on smaller scales this isn’t quite true. Galaxies and other clusters are not homogeneous and isotropic at all. A question that now immediately arises: where did the density fluctuations originate to form galaxies?
The cosmic microwave background (CMB) shows the afterglow of the Big Bang, with slight temperature differences indicating density fluctuations responsible for the structure in our universe. When cosmic strings were first predicted, they were a candidate for the seeds of density perturbations that cause the formation of structure. Recent studies have indicated that although cosmic strings could still be partly responsible for the density perturbations, Cosmological Inflation theory gives a better explanation for the current data retrieved from the CMB radiation. Therefore most interest in cosmic strings was lost.

Recently the interest in cosmic strings has been shifted to a more theoretical perspective. All Grand Unifying Theories predict the existence of cosmic strings. The observation of a cosmic string would validate one of the fundamental theories that predict strings.

Cosmic strings have certain remarkable properties that could have caused them to survive to the present day. There are several ways to detect cosmic strings. When a cosmic string is formed during one of the phase transitions, it creates a wedge in space-time. Light that passes a cosmic string is deformed due to the gravity exerted by the string. This effect is observable as the lensing of a cosmic string. The lensing causes two exact similar objects to appear in the sky.

The second way of detecting a cosmic string is by measuring gravitational waves. Cosmic strings are capable of emitting gravitational radiation by forming cusps, kinks or loops. These configurations can be formed when strings intersects with themselves or each other. Cusps emit radiation in a specific direction and with a relatively narrow spectrum. Loops decay by emitting a rather broad spectrum in a wide direction. Gravitational waves emitted by loops can therefore be more easily detected. Both types of emitted gravitational waves can in principle be detected, but that depends on the tension of the string. Cosmic strings have, until recently, not been found.

This thesis will give a review of past and current research on cosmic strings.

1.1 Outline

After a short introduction of symmetry breaking in chapter 2 this thesis will continue with the Abelian-Higgs model to explain the behaviour of cosmic strings. With the use of a small recap of General Relativity, the concept of cosmic lensing will be explained. Starting with the effective action of a cosmic string, the energy-momentum tensor will be derived in chapter 4. The Gauss-Bonnet theorem will be used to derive the wedge created in space-time that causes gravitational lensing.
More will be explained about cusps and kinks, which are sources of gravitational waves. The evolution of a cosmic string network and its scaling behaviour is mentioned in the final section of chapter 5. The thesis concludes with the contemporary status of observational evidence of the cosmic strings, including possible discoveries and recommended directions for future research.

1.2 Conventions

Throughout the thesis Einstein’s summation convention will be used:

$$\sum_{\mu, \nu=0}^{3} g_{\mu \nu} x^\mu x^\nu = g_{\mu \nu} x^\mu x^\nu$$ (1)

where the summation runs over \( \mu, \nu = 0, 1, 2, 3 \), where the zero indices is reserved for the time-component and the other three indices are space-like components. The following abbreviation will be used:

$$g = \det (g_{\mu \nu})$$ (2)

In the case of a flat universe: \( g_{\mu \nu} = \eta_{\mu \nu} \), where

$$\eta_{\mu \nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is called the a mostly-plus metric. Furthermore Planck’s constant and the speed of light have been set to one:

$$\hbar = c = 1$$ (3)

unless indicated otherwise.
2 Symmetry breaking

In this chapter topological defects will be explained. The first model we will use to describe a cosmic string, posesses a global symmetry which holds for all points in space-time. Defects of different dimensions, like domain walls and monopoles, are briefly mentioned and the Abelian-Higgs model is introduced. The Abelian-Higgs model exhibits a local symmetry. This means that the symmetry transformations can be different in different points of space-time. This forms the basis for gauge theories that will help us to avoid some of the problems encountered when discussing cosmic string models possessing a global symmetry.

2.1 Topological Defects

A topological defect can be compared to a ball balancing on top of a mountain. At a certain moment in time the ball will start to roll down and will eventually come to a stop at the foot of the mountain. The valley can be compared to a certain vacuum state, this state is said to be degenerate because there is a nonzero false vacuum state at \( x = 0 \) and (two) states equal to zero (see figure 1). To explore this in more detail, let’s look at the simplest example of spontaneous symmetry breaking. Consider a complex scalar field \( \phi \), sometimes called a Higgs field, which has the Lagrangian density [1]:

\[
L = g^{\mu\nu} \partial_\nu \phi^* \partial_\mu \phi - V(\phi, \phi^*)
\]

where the potential is given by:

\[
V(\phi, \phi^*) = \frac{1}{2} \lambda (|\phi|^2 - \frac{1}{2} \eta^2)^2
\]

The self-interaction term is denoted by \( \lambda \), it states how strongly two scalar particles interact and \( \eta \) is the mass term. This potential is also known as the Mexican hat potential (fig. 1). The Lagrangian has a rotational symmetry. This means that under (circular) transformations in the \( \phi, \phi^* \) plane the Lagrangian does not change. Just by looking at figure 1 it can be seen that the potential is invariant under U(1) transformations: the shape of the potential does not change, no matter in what direction you look at the x-y plane. The transformation

\[
\phi \rightarrow \phi e^{i\alpha}
\]

is unitary, since \( e^{i\alpha}e^{-i\alpha} = 1 \). When we apply this transformation to the Lagrangian (4):

\[
L = g^{\mu\nu} \partial_\nu \phi^* e^{-i\alpha} \partial_\mu \phi e^{i\alpha} - V(\phi, \phi^*) = g^{\mu\nu} \partial_\nu \phi^* \partial_\mu \phi - V(\phi, \phi^*)
\]
The dynamics of a real scalar field is described by the action:

$$ S[\phi] = \int dt L[\phi] = \int dt \int d^3x L(\phi, \partial_\mu \phi, \dot{\phi}) $$  \hspace{1cm} (8)

Hamilton’s extremal principle states that the configurations $\phi$ that are actually realized are those that extremize the action, i.e. the action is stationary under small variations of the fields:

$$ S[\phi + \delta \phi] = S + \delta S $$  \hspace{1cm} (9)

The stationary action principle states that $\delta S = 0$. This means that for any smooth curve:

$$ \lim_{\epsilon \to 0} \frac{1}{\epsilon} (S[\phi + \epsilon \theta] - S[\phi]) = 0 $$  \hspace{1cm} (10)

This becomes for the action:

$$ \delta S = \int d^nx [L(\phi + \epsilon \theta, \partial_\mu \phi + \epsilon \partial_\mu \theta) - L(\phi, \partial_\mu \phi)] $$

$$ = \int d^nx [\partial_\theta \frac{\partial L}{\partial \phi} + \frac{\partial L}{\partial (\partial_\mu \phi)} \partial_\mu \theta] \epsilon + O(\epsilon^2) $$

$$ = \int d^nx [\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)}] \theta \epsilon + O(\epsilon^2) $$  \hspace{1cm} (11)

In order to set $\delta S = 0$ we need:

$$ \frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} = 0 $$  \hspace{1cm} (12)
In our case, the Lagrangian contains a complex field that can be written as two real fields, so there should be an equation for each field [3]:

$$\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} = 0, \quad \frac{\partial L}{\partial \phi^*} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi^*)} = 0$$

(13)

This is known as the Euler-Lagrange equation [4]. The first term of the Euler-Lagrange equation is given by:

$$\frac{\partial L}{\partial \phi} = -\frac{\partial V}{\partial \phi} = -\partial \frac{1}{2}\lambda(|\phi|^2 - \frac{1}{2}\eta^2)^2)$$

$$= -\lambda(|\phi|^2 - \frac{1}{2}\eta^2)\phi^*$$

(14)

And the second term is given by:

$$-\partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} = -\partial_\mu \partial^\mu \phi = -\partial^2 \phi^*$$

(15)

The same equations are found for the real scalar field, which gives the following field equations:

$$\partial^2 \phi^* + \lambda(|\phi|^2 - \frac{1}{2}\eta^2)\phi^* = 0, \quad \partial^2 \phi + \lambda(|\phi|^2 - \frac{1}{2}\eta^2)\phi = 0$$

(16)

The second equation is found by applying complex conjugation. The Higgs-boson can be derived in a somewhat similar expression, as can be seen in appendix A. When we look at the extrema of the potential, there is a solution obtained by solving \(\phi = 0\), sometimes called the false vacuum. The real vacuum is at:

$$\delta V \delta \phi = \lambda(|\phi|^2 - \frac{1}{2}\eta^2)\phi = 0$$

(17)

which gives for \(\phi\):

$$|\phi|^2 = \frac{1}{2}\eta^2$$

(18)

So the minima of the potential lie on a circle (according to global U(1) symmetry). The field can be continuously changed by rotating through the complex field, staying always in a state of minimal potential energy:

$$\phi(\theta) = \frac{1}{\sqrt{2}}\eta e^{i\theta}$$

(19)

where \(e^{i\theta}\) is an arbitrary phase constant. Any phase of the field describes a vacuum state. But the ground states do not have the same symmetries as the Lagrangian anymore. Under a gauge transformation a state transforms into:

$$\phi(\theta) e^{i\alpha} = \frac{1}{\sqrt{2}}\eta e^{i\theta} e^{i\alpha}$$

(20)
The symmetry is spontaneously broken. The string configuration arises from potential and gradient contributions to the energy density, which can be derived as follows: a large circle can be drawn in space with the complex phase vectors all pointing outwards (this description is only valid for \( n = 1 \)). All these points have the same energy (the energy of the ground state). Somewhere in the middle the field must pass through \( \phi = 0 \), implying the existence of an excitation carrying energy. This excitation is stable and called a cosmic string. The point inside this loop has energy higher than the ground state. In polar coordinates:

\[
\phi(\theta) = \frac{1}{\sqrt{2}} e^{in\theta}
\]  (21)

where \( n \) is an integer. So what is the energy-density of such a configuration? We are interested in solutions matching specified boundary conditions at infinity \( (r \to \infty) \) [1]. The gradient in polar coordinates is given by [5]:

\[
\nabla \phi = \frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \hat{\varphi}
\]  (22)

Only the second term contributes, because \( \phi \) only depends on \( \theta \):

\[
\nabla \phi(\theta) = \frac{1}{r} \frac{1}{\sqrt{2}} e^{in\theta} \hat{\theta} = \frac{in}{r} \phi(\theta) \hat{\theta}
\]  (23)

Consider the energy density for a static configuration:

\[
\mathcal{H} = \pi \dot{\phi} - \mathcal{L}
\]  (24)

where \( \pi = \frac{\partial L}{\partial \dot{\phi}} \). In a static configuration the energy density is equal to \( -\mathcal{L} \) because the time-dependent factors do not contribute:

\[
\mathcal{H} = -\mathcal{L} = -g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + V(\phi) = -|\nabla \phi|^2 + V(\phi)
\]  (25)

For \( r \to \infty \), the potential term \( V(\phi) \) goes to zero, but the squared kinetic term is proportional to \( \frac{1}{r^2} \). The total energy can be found by integrating over the Lagrangian density [6]:

\[
\mathcal{E} = -\int \! d^2 x \; \mathcal{L} = \int \frac{1}{r^2} r dr d\theta
\]  (26)

Integrating over \( d\theta \) will give a contribution of \( 2\pi \). Integrating over \( dr \) will give an infinite contribution to the energy density. It is not possible to prevent this outcome from happening. This coincides with Derrick’s theorem (see Appendix B). But it can be evaded by adding an extra term to the Lagrangian.
2.2 Abelian Higgs model

The symmetry described above is a global symmetry. Their symmetry transformations involve rotating every point in the field by the same constant. A local symmetry allows each point to vary by a different angle. This symmetry is present in the Abelian Higgs model described below. The Lagrangian of the previous example led to an infinite energy density contribution. In the case of the Abelian-Higgs model, a gauge field $A_\mu$ will cancel out the divergences, which will lead to an energy density that will no longer be infinite, as happened in the previous section. In this case, the Lagrangian equals [7]:

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - V(\phi, \phi^*)$$  \hspace{1cm} (27)

where the potential is the same as the previous example, $D_\mu = \partial_\mu - ieA_\mu$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength. The local invariance is realized by:

$$\phi \rightarrow e^{i\alpha(x)}\phi(x) \hspace{1cm} (28)$$

$$A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu \alpha(x) \hspace{1cm} (29)$$

Deriving the field equations we again need the Euler-Lagrange equations [1] and the complex conjugates:

$$\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} = 0, \hspace{1cm} \frac{\partial L}{\partial \phi^*} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi^*)} = 0;$$  \hspace{1cm} (30)

$$\frac{\partial L}{\partial A_\mu} - \partial_\nu \frac{\partial L}{\partial (\partial_\nu A_\mu)} = 0, \hspace{1cm} (31)$$

Since $F_{\mu\nu}$ does not depend on $\phi$ or $\phi^*$, the first field equation (30) will be the same as before:

$$D^2 \phi + \lambda(\phi^2 - \frac{1}{2}\eta^2)\phi = 0, \hspace{1cm} D^2 \phi^* + \lambda(\phi^2 - \frac{1}{2}\eta^2)\phi^* = 0 \hspace{1cm} (32)$$

The first term of the second field equation (31) equals:

$$\frac{\partial L}{\partial A_\mu} = \frac{\partial}{\partial A_\mu} |D_\mu \phi|^2 = \frac{\partial}{\partial A_\mu} (\partial_\mu - ieA_\mu)(\partial^\mu + ieA^\mu)\phi^*$$

$$= \frac{\partial}{\partial A_\mu} (\partial_\mu \phi \partial^\mu \phi^* - ieA_\mu \phi \partial^\mu \phi^* + \partial_\mu \phi ieA^\mu \phi^* - i^2 e^2 A_\mu A^\mu \phi^* \phi)$$

$$= -ie\phi \partial^\mu \phi^* + \partial_\mu \phi ie\phi^* - i^2 e^2 A^\mu \phi^* \phi + -i^2 e^2 A_\mu \phi^* \phi$$

$$= ie(\phi^* D^\mu \phi - (D^\mu \phi^*)\phi) \hspace{1cm} (33)$$

15
since only $|D_\mu \phi|^2$ can be functional differentiated with respect to $A_\mu$. For the second term only $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ depends on $\partial_\nu A_\mu$:

$$\partial_\nu \frac{\partial L}{\partial (\partial_\nu A_\mu)} = -\partial_\nu \frac{\partial}{\partial (\partial_\nu A_\mu)} \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial_\nu A^\mu)$$

$$= -\partial_\nu \frac{8}{4} (-\partial^\mu A^\nu + \partial_\nu A^\mu) = 2 \partial_\nu F^{\mu\nu}$$  \hspace{1cm} (34)

The two terms combined give:

$$ie(\phi^* D_\mu \phi - D_\mu \phi^* \phi) - 2 \partial_\nu F^{\mu\nu} = 0$$ \hspace{1cm} (35)

The gauge field is constructed in such a way that it cancels out the derivative of the gradient term from our previous example:

$$A = \frac{1}{e} \nabla(n\theta) = \frac{1}{e} \nabla(n\theta) + \frac{1}{e} n \nabla(\theta) = 0 + \frac{n}{e} \frac{1}{r} \frac{\partial \theta}{\partial \theta} = \frac{n}{r}$$  \hspace{1cm} (36)

as $r \to \infty$. The total contribution to the kinetic energy is now given by:

$$|D_\mu \phi|^2 = |(\nabla_\mu - ieA_\mu)\phi|^2$$ \hspace{1cm} (37)

Keeping only static contributions and using (23):

$$D_r \phi = (\nabla_r - ieA_r)\phi = \frac{in}{r} \phi - \frac{in}{r} \phi = 0;$$ \hspace{1cm} (38)

$$D_\theta \phi = \frac{1}{r} (\nabla_r - ieA_\theta)\phi = 0$$ \hspace{1cm} (39)

The kinetic term no longer gives an infinite contribution to the energy density. In this case it gives a finite contribution to the energy density [1]:

$$\mathcal{H} = -\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + V(\phi, \phi^*)$$ \hspace{1cm} (40)

Thus the kinetic term goes to zero in the Abelian Higgs model. The field tensor $F^{\mu\nu}$ is written in terms of the four-vector potential [5]:

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$ \hspace{1cm} (41)

where $A = (V, A_x, A_y, A_z)$. The Maxwell equations, defined up to a gauge transformation, are:

$$\mathbf{B} = \nabla \times \mathbf{A};$$ \hspace{1cm} (42)

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$ \hspace{1cm} (43)
The static contribution of the Maxwell equations is:

\[-\frac{1}{4} F_{\mu \nu} F^{\mu \nu} = -\frac{1}{4} (\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i) - \frac{1}{4} (\partial_0 A_i - \partial_i A_0)(\partial^0 A^i - \partial^i A^0)\]

\[= -\frac{1}{4} (\nabla \times A)^2 - (\frac{\partial A}{\partial t} - \nabla \phi)^2 = -\frac{1}{4} (B^2 + E^2)\] (44)

So the total contribution to the static energy density is:

\[\mathcal{H} = \frac{1}{4} (B^2 + E^2) + V(\phi)\] (45)

At large r, the energy density goes to zero, since the electric and magnetic field components do not contribute. This solution carries a magnetic flux. This can be seen when considering the flux through an area around the cosmic string: \[\Phi = \int \mathbf{B} \cdot d\mathbf{a}.\] According to Stokes’ theorem [5] and using (36):

\[\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \int \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l} = \oint -\mathbf{n}rd\theta = 2\pi n e\] (46)

Thus the magnetic flux caused by the string is quantized [6]. In Appendix A is shown how the Higgs-model is related to the Abelian Higgs model.

### 2.3 Cosmic strings in the universe (Kibble mechanism)

The Kibble mechanism states that when a topological defect can form at a phase transition, it will form. This statement is based on the argument that different regions in space cannot communicate with each other. In the early universe causal effects propagated with the speed of light. Different regions separated by more than a distance \(d = ct\) do not know anything about each other. As stated before, \(c = 1\) according to the conventions used throughout this thesis. This distance is also called the causal horizon [6]. This can be written as:

\[\xi \leq t\] (47)

where \(\xi\) is the correlation length, defined by the distance of possible interaction and \(t\) is the age of the universe. At some point there was a phase transition, that’s when different regions would pick different phases that leads to the cosmic string. The strings that were formed grew as the universe expanded, giving rise to infinite long strings [18] and were able to form a string network. During further development of the universe, the strings were able to interact and form loops. More about string networks and string interaction will be discussed in chapter 5.
2.4 Monopoles, Textures and Domain walls

Cosmic strings are two-dimensional topological defects. There can also appear topological defects of other dimensions, causing monopoles and textures to arise. The type of defect that is produced depends on the symmetry that has been broken [8]. Although other topological defects like domain walls, monopoles and textures are not the main focus of this thesis, I will give a short introduction. Where \( n \) are the dimensions orthogonal to the defect:

- \( n = 1 \) Domain walls: different regions in space in one of the two phases are separated by a domain wall. The universe is divided in discrete cells. This happens when a discrete symmetry is broken.

- \( n = 2 \) Strings: arise when the symmetry of a circle is broken, as the Lagrangian of the beginning of this chapter.

- \( n = 3 \) Monopoles: are point-like defects, with a specific magnetic charge, either a north or a south pole. These types of defects arise when spherical symmetry has been broken.

- \( n = 4 \) Textures: are formed when more complex symmetries are broken. These types of defects are unstable.

When a discrete symmetry is broken at a phase transition, there is a possibility that two-dimensional domain walls appear. The domain walls will divide the universe into different areas. An remarkable property of a domain wall is that the gravitational field is repulsive rather than attractive [6]. A universe that consists of domain walls will look something like this:

![Figure 2: Domain walls [9].](image)
Another topological defect is a zero-dimensional or point-like object called a monopole. This defect occurs when a spherical symmetry is broken. They are predicted by the grand unified theories (GUTs) and are supposed to be super massive and carry magnetic charge. They remain one of the problems of standard cosmology, as none have been observed. Cosmological Inflation resolves this problem by exponential reduction in density due to the exponential expansion of the universe.

![Magnetic monopole](image1)

Figure 3: Magnetic monopole [9].

Textures will form when more complicated symmetries are broken. They are unstable and able to unwind. Their configuration in one and two dimensions will look something like this:

![Texture in one dimension (a) and two dimensions (b)](image2)

Figure 4: Texture in one dimension (a) and two dimensions (b) [9].
3 A little bit of General Relativity

In this section I will introduce some basic details of General Relativity. The theory of General Relativity, defined by Einstein in 1916, is a generalization of the theory of Special Relativity, also introduced by Einstein in 1905. General Relativity describes a relativistic theory of gravity. The concept of a metric, the use of geodesics and applying the Einstein equations to the Friedmann-Robertson-Walker metric are the main topics we will encounter hereafter. For more background information on these subjects I refer to Spacetime and Geometry by Carroll [10], [11] and Modern Cosmology by Dodelson [12].

3.1 The metric

The metric turns coordinate distance into proper time. This proper time is invariant: it does not depend on the observer. The metric includes the effects of gravity so that particles can move freely in a curved space-time instead of regarding gravity as an external force. We can say that the curvature of space-time is a manifestation of gravity. In the next paragraph we will discover that the so-called Einstein equations are the outcome of how space-time curvature responds to the presence of matter and energy (and thus gravity). The line element, also called the proper time, is given by:

\[ ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu \] (48)

The line element describes the infinitesimal length of the path of a particle moving through space-time. Special relativity is described by Minkowski space-time with the metric:

\[ g_{\mu\nu} = \eta_{\mu\nu}, \] with:

\[ \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

In Cosmology, the metric is not of Minkowski type to allow for the expansion of the universe. There are several scenarios concerning the expansion of the universe: the spatial geometry can be closed, flat or open. These different types of geometries can be visualized by considering two freely moving particles travelling parallel through space-time. When the universe is closed, the particles will gradually move towards each other. When the universe is flat the particles will keep travelling parallel to each other. The geometry of an open universe causes the particles to move further away from each other. We will assume that the space-time geometry is flat, in
line with observations. The metric in an expanding, flat universe (Friedmann-Robertson-Walker (FRW) metric) reads:

\[
g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & a^2(t) & 0 & 0 \\
0 & 0 & a^2(t) & 0 \\
0 & 0 & 0 & a^2(t)
\end{pmatrix}
\]

The scale factor \(a(t)\) is the co-moving distance between coordinates [12]: the distance between the coordinates is proportional to the scale factor. At early times, during the beginning of the universe, the distance scaled according to:

\[a(t) \propto \sqrt{t}\]  \hspace{1cm} (49)

where \(t\) is the cosmic time.

### 3.2 The geodesic

The geodesic is the path followed by any particle in the absence of any forces. To express this in equations, we must generalize Newton’s second law with no forces: \(F = \frac{d^2\xi}{dt^2} = 0\) in the case of curved space-time. First I shall introduce the Principle of Equivalence. This principle states that "In small enough regions of space-time, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments" [10]. Now consider a free-falling particle. According to the Einstein Equivalence Principle (EEP) there is no external force exerted on the particle, reducing Newton’s second law to:

\[\frac{d^2\xi^\alpha}{d\tau^2} = 0\]  \hspace{1cm} (50)

The solution of this equation is given by: \(\xi^\alpha(\tau) = a^\alpha \tau + b^\alpha\), a straight line in space-time. Now consider a different coordinate system: \(\xi^\mu : \xi^\alpha = \xi^\alpha(x^\mu)\). These coordinates could still be Cartesian, but can also rotate, be curved or accelerated with regard to our original coordinate system. The equation above becomes:

\[\frac{d}{d\tau} \left( \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{dx^\mu}{d\tau} \right) = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\]  \hspace{1cm} (51)

Multiplying both sides by \(\frac{\partial x^\lambda}{\partial \xi^\alpha}\) gives:

\[\delta_\mu^\lambda \frac{d^2x^\mu}{d\tau^2} + \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{d^2\xi^\alpha}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\]  \hspace{1cm} (52)
introducing the Christoffel symbol, defined as: \[ \Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \], gives the equation of motion:

\[
\frac{d^2 x^\lambda(\tau)}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu(\tau)}{d\tau} \frac{dx^\nu(\tau)}{d\tau} = 0 \tag{53}
\]

This equation of motion can be used in a curved, accelerated or rotating coordinate system. In a flat geometry, non-accelerating or non-rotating system, the Christoffel symbol is equal to zero and the equation above is reduced to (50).

### 3.3 Einstein equations

The Einstein equation relates components of the Einstein tensor to the energy-momentum tensor. It "governs" how the metric responds to energy and momentum. This field equation must be postulated, but can be derived from some plausible arguments. We would like to find a relativistic generalization of the Poisson equation of the Newtonian potential:

\[
\nabla^2 \Phi = 4\pi G \rho \tag{54}
\]

Where \( \nabla^2 \) is the Laplacian and \( \rho \) the mass density. To transform this equation into a relativistic form, let’s look at the current characteristics first. On the left-hand side a Laplacian is acting on the gravitational potential, on the right-hand side there is the mass distribution. The generalization of the mass distribution is the energy-momentum tensor \( T_{\mu\nu} \). The gravitational potential should be replaced by a metric tensor. Unfortunately, it’s not just the metric tensor. Instead we need to construct the appropriate tensor from the metric and the derivatives of the metric: the Riemann tensor. The Ricci tensor in terms of the Christoffel symbol is described as:

\[
R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha} \Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu} \Gamma^\beta_{\mu\alpha} \tag{55}
\]

where the commas are the derivatives with respect to \( x \). The Christoffel symbol in terms of the metric is described as:

\[
\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2} \left[ \frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right] \tag{56}
\]

where \( g^{\mu\nu} \) is the inverse of \( g_{\mu\nu} \). The Ricci tensor contains the curvature of space-time. A first guess could be to generalize the Newtonian potential by plugging in the Ricci-tensor. If we want to preserve conservation of energy \( \nabla^\mu T_{\mu\nu} = 0 \), the field equations would imply:

\[
\nabla^\mu R_{\mu\nu} = 0 \tag{57}
\]

This is equation is not true. We do know a symmetric tensor, constructed from the Ricci tensor, which is automatically conserved: the Einstein Tensor. The Einstein tensor is automatically conserved due to the Bianchi identity:

\[
R_{\mu\nu;\alpha} + R_{\lambda\mu\nu;\alpha} + R_{\lambda\mu\nu;\alpha} = 0 \tag{58}
\]
we have \( R_{\mu\nu} = g^{\lambda\kappa} R_{\lambda\mu\nu\kappa}, \ R_{\mu\kappa} = R^\lambda_{\mu\lambda\kappa} \) and \( R_{\lambda\mu\nu\kappa} = -R_{\lambda\mu\nu\kappa}. \) Multiplying (58) first by \( g^{\lambda\kappa} \) and secondly with \( g^{\mu\nu} \) gives the desired result:

\[
g^{\mu\nu}(g^{\lambda\kappa} R_{\lambda\mu\nu\kappa;\eta} + g^{\lambda\kappa} R_{\lambda\mu\nu\kappa;\eta} + g^{\lambda\kappa} R_{\lambda\mu\nu\kappa}) = 0 \tag{59}
\]

\[
g^{\mu\nu} R_{\mu\nu;\eta} - g^{\mu\nu} R_{\mu\eta;\nu} + g^{\mu\nu} R^{\kappa}_{\mu\nu;\kappa} = 0 \tag{60}
\]

\[
R_{\eta} - R^{\eta}_{\eta\nu} - R^{\kappa}_{\kappa\nu} = 0 \tag{61}
\]

This is known in a more well known form as [13]:

\[
(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)_{;\mu} = 0 \tag{62}
\]

Our field equation is now of the form:

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \propto 4\pi G T_{\mu\nu} \tag{63}
\]

where \( G \) corresponds to Newton’s gravity constant. The Einstein equation is equal to:

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{64}
\]

where \( G_{\mu\nu} \) is the Einstein tensor, \( R_{\mu\nu} \) is the Ricci Tensor, and \( R \) is the Ricci scalar: the contraction of the Ricci tensor \( (R \equiv g^{\mu\nu} R_{\mu\nu}) \), \( G \) is Newton’s constant and \( T_{\mu\nu} \) the energy-momentum tensor. Einstein’s field equation describes how the metric responds to energy and momentum. The field equation is postulated. The energy-momentum tensor contains the energy and momentum of matter; the Einstein equation relates energy to curvature.

### 3.4 Einstein gravity of a string

In this section the Einstein equations of a cosmic string will be derived. A string in Minkowski space lying along the z-axis (fig. 5) is invariant under certain translations and rotations: time translations, spatial translations in the z-direction, rotations around the z-axis and Lorentz boost in the z-direction.

The reasonable assumption is made that the space-time of a gravitating string has the same symmetries. The metric in terms of cylindrical coordinates can now be written as [6]:

\[
d s^2 = d t^2 - d r^2 - d z^2 - C^2(r) d \theta^2 \tag{65}
\]

in matrix form:

\[
d s^2 = g_{\mu\nu} d x^\mu d x^\nu = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -C^2(r)
\end{pmatrix} d x^\mu d x^\nu
\]
Only the $g_{\theta\theta} = C(r)^2$ depends on $r$, and: $\theta \subset [0, 2\pi]$. When $C^2 = r^2$ the metric describes flat space-time, but when it is proportional to $Ar^2$, a slice is taken out of space-time. We assume that the energy-momentum tensor is reduced to:

$$T_{\mu\nu} = \mu\delta(x)\delta(y) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ (66)

The energy of the energy-momentum tensor is aligned along the $z$-direction. Later on we will more or less derive this expression from the effective action $S$. There will be only a contribution to the Einstein equations (64) of the $T_{tt}$ and $T_{zz}$ components, so they will reduce in this case to:

$$R_{tt} - \frac{1}{2} g_{tt} \mathcal{R} = 8\pi G T_{tt}$$ (67)

$$R_{zz} - \frac{1}{2} g_{zz} \mathcal{R} = 8\pi G T_{zz}$$ (68)

The Einstein equations are given in terms of the Ricci tensor and the Ricci tensor can be expressed in terms of the Christoffel symbol (55). As we can easily see from expression (56), the only contribution to the Christoffel symbol are coming from the $\theta\theta$-components. The Christoffel symbol can be described in terms of the metric (56). This component is the only term depending on $r$. The other terms do
not depend on \( r \), or on any other variable, and will give a zero contribution when differentiated. There are three terms contributing to the Christoffel symbol:

\[
\Gamma^\theta_{\theta r} = g^{\theta\theta} \left( \frac{\partial g_{\theta\theta}}{\partial r} \right) = \frac{1}{2} \frac{\partial C^2}{\partial r} \frac{1}{1} 2CC' = C''/C; \tag{69}
\]

\[
\Gamma^\theta_{r\theta} = g^{\theta\theta} \left( \frac{\partial g_{\theta\theta}}{\partial r} \right) = C'/C; \tag{70}
\]

\[
\Gamma^r_{\theta\theta} = -\frac{g^r}{2} \left( \frac{\partial g_{\theta\theta}}{\partial r} \right) = -\frac{1}{2} 2CC'' = -C'C \tag{71}
\]

where the primes denote differentiation with respect to \( r \). There are only three contributions to the Christoffel symbol that are unequal to zero. The two contributions to \( R_{rr} \) are given by:

\[
R_{rr} = \Gamma^\alpha_{rr,\alpha} - \Gamma^\alpha_{\alpha r, r} + \Gamma^\alpha_{\beta r} \Gamma^\beta_{rr} = -\Gamma^\theta_{r\theta,r} - \Gamma^\theta_{r\theta, r} \tag{72}
\]

The second term in the first line cancels, and the fourth term has two contributions leading to the cancellation of the second and third term against each other in the second line. The Ricci scalar is given by:

\[
R = g^\mu\nu R_{\mu\nu} = g^{tt} R_{tt} + g^{rr} R_{rr} + g^{zz} R_{zz} + g^{\theta\theta} R_{\theta\theta} \tag{74}
\]

At the second equality we used the fact that the components of the Ricci tensor that do not depend on \( r \) or \( \theta \) are zero. The Ricci scalar is equal to:

\[
R = g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} = \frac{C''}{C} + \frac{-1}{C^2} (-C'C) = \frac{C''}{C} + \frac{C''}{C} = 2\frac{C''}{C} \tag{75}
\]

We already know that the \( R_{tt} \) and \( R_{zz} \) components are equal to zero because they do not contain any \( r \)-dependence. The Einstein equations simple reduce to:

\[
R_{tt} - \frac{1}{2} g_{tt} R = -\frac{1}{2} (2 \frac{C''}{C}) = -8\pi G\mu; \tag{76}
\]

\[
R_{zz} - \frac{1}{2} g_{zz} R = -\frac{1}{2} (-1)(2 \frac{C''}{C}) = 8\pi G\mu \tag{77}
\]
We can conclude that the Einstein equations, with the use of (66), have been reduced to [6]:

$$\frac{C''}{C} = 8\pi G \mu$$

(78)

In the final paragraph of the next chapter this result will be derived in a different manner. The result will be used to derive the wedge that has been removed from space-time by the presence of the cosmic string.
4 Cosmic String as a Source

During this chapter and the next we will go into more detail about possible observations or detection of cosmic strings. There are several ways to perceive a cosmic string. The final paragraph of this chapter will give more insight in gravitational lensing by a cosmic string. In order to fully comprehend the theorem necessary to derive the lensing angle, the effective cosmic string action and the energy-momentum tensor of a cosmic string will be derived.

4.1 The action formalism of gravity

Following Lecture Notes on General Relativity by Carrol [11] we will derive the Einstein equation from the action principle. Re-deriving the Einstein equations from the effective action will show that the Lagrangian $\mathcal{L}$ used in chapter 2 is a suitable choice. In the case of the Lagrangian formulation the Einstein equations can be derived from:

$$S_H = \int d^nx \mathcal{L}_H$$

(79)

The Lagrangian density is written as: $\mathcal{L} = \sqrt{-g} R$, where $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar and $g = \det(g_{\mu\nu})$. The action reads:

$$S_H = \int d^nx \sqrt{-gg^{\mu\nu} R_{\mu\nu}}$$

(80)

This is the simplest choice for the Lagrangian. This is the only non-trivial scalar constructed in terms of the metric, which has no higher than second order derivatives. As we have seen before (10) the equations of motion come from varying the action with respect to the metric:

$$\delta S = \int d^nx \left[ \sqrt{-g}g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + R \delta \sqrt{-g} \right]$$

(81)

$$= (\delta S)_1 + (\delta S)_2 + (\delta S)_3$$

(82)

The second term is already of a form that can be used directly. The first and third term have to be rewritten. Starting with $(\delta S)_1$, for arbitrary variations of the Ricci tensor, which can be expressed in terms of the Christoffel symbol (55):

$$\Gamma^\rho_{\nu\mu} \rightarrow \Gamma^\rho_{\nu\mu} + \delta \Gamma^\rho_{\nu\mu}$$

(83)

When taking the covariant derivative:

$$\nabla_\lambda (\delta \Gamma^\rho_{\nu\mu}) = \partial_\lambda (\delta \Gamma^\rho_{\nu\mu}) + \Gamma^\rho_{\lambda\sigma} \delta \Gamma^\sigma_{\nu\mu} - \Gamma^\rho_{\lambda\nu} \delta \Gamma^\sigma_{\mu\sigma} - \Gamma^\rho_{\lambda\mu} \delta \Gamma^\sigma_{\nu\sigma}$$

(84)

$$\nabla_\nu (\delta \Gamma^\rho_{\lambda\mu}) = \partial_\nu (\delta \Gamma^\rho_{\lambda\mu}) + \Gamma^\rho_{\nu\sigma} \delta \Gamma^\sigma_{\lambda\mu} - \Gamma^\rho_{\nu\lambda} \delta \Gamma^\sigma_{\mu\sigma} - \Gamma^\rho_{\nu\mu} \delta \Gamma^\sigma_{\lambda\sigma}$$

(85)
It can be shown that for \((\delta S)_1:\)

\[
\delta R_{\mu\nu} = \partial_\lambda (\delta \Gamma^\lambda_{\nu\mu}) + \delta \Gamma^\rho_{\lambda\nu} \Gamma^\sigma_{\nu\mu} + \Gamma^\rho_{\lambda\sigma} \delta \Gamma^\sigma_{\nu\mu} - \partial_\nu (\delta \Gamma^\lambda_{\lambda\mu}) - \delta \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\lambda\mu} - \Gamma^\rho_{\nu\sigma} \delta \Gamma^\sigma_{\lambda\mu}
\]

\[= \nabla_\lambda (\delta \Gamma^\lambda_{\nu\mu}) - \nabla_\nu (\delta \Gamma^\lambda_{\lambda\mu}) \quad (86)\]

Because the Christoffel symbol is symmetric in the two lower indices, the third term in (84) and (85) will disappear when subtracted. When some dummy-indices are relabelled the first contribution can be written as:

\[
(\delta S)_1 = \int d^n x \sqrt{-g} g^{\mu\nu} [\nabla_\lambda (\delta \Gamma^\lambda_{\nu\mu}) - \nabla_\nu (\delta \Gamma^\lambda_{\lambda\mu})]
\]

\[= \int d^n x \sqrt{-g} [g^{\mu\nu} \nabla_\lambda (\delta \Gamma^\lambda_{\nu\mu}) - g^{\mu\nu} \nabla_\nu (\delta \Gamma^\lambda_{\lambda\mu})] \quad (87)\]

\[= \int d^n x \sqrt{-g} \nabla_\sigma [g^{\mu\sigma} (\delta \Gamma^\lambda_{\lambda\mu}) - g^{\mu\nu} (\delta \Gamma^\sigma_{\nu\mu})] \quad (88)\]

The equivalence principle demands that: \(\nabla_\sigma g^{\mu\sigma} = 0,\) now (87) can be written as (88). The result is an integral with respect to the natural volume element of the covariant divergence of a vector. According to Stokes’ theorem this is equal to the boundary contribution at infinity. We can set this to zero by making the variation vanish at infinity. Therefore the first term does not contribute to the total variation. Looking at the \((\delta S)_3,\) the following identity can be applied for any matrix \(M:\)

\[Tr(\ln M) = \ln(\det M) \quad (89)\]

The variation of this identity gives:

\[Tr(M^{-1} \delta M) = \frac{1}{\det M} \delta(\det M) \quad (90)\]

In this case: \(M = g^{\mu\nu},\) inserting \(M = g^{\mu\nu}\) in the equation above, then \(\det M = g^{-1}\) and:

\[\delta(g^{-1}) = \frac{1}{g} g_{\mu\nu} \delta g^{\mu\nu} \quad (91)\]

Now with the use of the equation above:

\[\delta(\sqrt{-g}) = \delta([-g^{-1}]^{-1/2}) = -\frac{1}{2} (-g^{-1})^{-3/2} \delta(-g^{-1})
\]

\[= -\frac{1}{2} (-g^{-1})^{-3/2} \frac{1}{-g} g_{\mu\nu} \delta g^{\mu\nu} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \quad (92)\]

And so \((\delta S)_3 = -\int d^n x \sqrt{-g} \frac{1}{2} R g_{\mu\nu} \delta g^{\mu\nu}.\) Since the first term \((\delta S)_1\) does not contribute, we are left with the second and third term:

\[\delta S = \int d^n x \sqrt{-g} [R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}] \delta g^{\mu\nu} \quad (93)\]
Since this must be true for arbitrary variations:

\[
\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0
\]  

(94)

these are the vacuum field equations. In the presence of matter, the total action should look like:

\[
S = \frac{1}{8\pi G} S_H + S_M
\]  

(95)

If we apply the variational principle again this will lead to:

\[
\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^\mu\nu} = \frac{1}{8\pi G} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) + \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g^\mu\nu} = 0
\]  

(96)

when we look at the Einstein equations, we can identify:

\[
T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^\mu\nu}
\]  

(97)

So by choosing the simplest form of the Lagrangian we have re-derived the Einstein equations. We know that the energy momentum tensor is symmetric and conserved. We can check what this formulation gives for varying the Lagrangian (4) and \( S = \int \sqrt{-g} \mathcal{L} \) with respect to the inverse metric:

\[
\frac{\delta S_M}{\delta g^\mu\nu} = \int d^4x [\sqrt{-g} \delta g^\mu\nu (\partial_\mu \phi \partial_\nu \phi^*) + \delta \sqrt{-g} (g^\mu\nu \partial_\rho \phi \partial_\sigma \phi^* - V(\phi, \phi^*) )]
\]  

(98)

As we have seen before: \( \delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^\mu\nu \). Inserting this equality into (98) gives:

\[
\frac{\delta S_M}{\delta g^\mu\nu} = \int \sqrt{-g} \delta g^\mu\nu [\partial_\mu \phi \partial_\nu \phi^* + (-\frac{1}{2} g_{\mu\nu}) (g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi^* - V(\phi, \phi^*) )]
\]  

(99)

\[
= -\sqrt{-g} T_{\mu\nu}
\]  

(100)

So the energy-momentum tensor equals:

\[
T_{\mu\nu} = -\partial_\mu \phi \partial_\nu \phi^* + \frac{1}{2} g_{\mu\nu} [g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi^* - V(\phi, \phi^*) ]
\]  

(101)

This result coincides with what we expected.

### 4.2 Effective Cosmic String Action

Consider a straight cosmic string in the z-direction. On physical grounds the string has a tension and its thickness can be neglected. The stress energy tensor
can therefore be approximated by (66). We now construct an effective action of a cosmic string, which will give us the expected energy-momentum tensor. The Nambu-action of a single particle in one dimension will be used to derive the action of a string in two dimensions, leading to the energy-momentum tensor in flat space-time of a cosmic string. The ”physics” of a single particle moving through space-time can be described by the simple action [6], [14]:

\[ S_{\text{particle}} = \int ds \]  

(102)

This will look like:

\[ x(t) \]

Figure 6: A particle moving through space-time.

where \( ds^2 \) is defined as usual by: \( ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 \). So (102) can be written as:

\[ S_{\text{particle}} = \int \sqrt{1 - g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \]  

(103)

The equations of motions are derived by taking the variation of the action with respect to \( x^\mu \): \( \frac{\delta S}{\delta x^\mu} = 0 \), as will be done in chapter 5. The simplest form of the action of a string is slightly more complicated. Since a string moves in two dimensions, it creates a world sheet in space-time (fig. 7). In this case our action will look like:

\[ S_{\text{string}} = -\mu \int d^2 \zeta \sqrt{-\gamma} \]  

(104)

where \( \mu \) has the dimension of the tension of the cosmic string. The action has now become a dimensionless quantity. In this case there is a different metric \( \gamma \), depending on the choice of or coordinate system. The function \( x^\mu \) maps every coordinate from flat space-time to curved space-time:

\[ x^\mu \rightarrow x^\mu(\zeta^a), \quad a = 0, 1 \]  

(105)
Figure 7: A moving string creating a world sheet in space-time.

Figure 8: Going to curved space-time.
Going from flat to curved space-time is visualized in fig. 8. The new metric defined in terms of the old metric is now given by:

\[ \gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \zeta^a} \frac{\partial x^\nu}{\partial \zeta^b} \]  

This is called the induced metric. We have now found the two dimensional world sheet metric. The string energy-momentum tensor can be found by varying the effective action with respect to the metric \( g_{\mu\nu} \):

\[ \delta S = -\mu \int d^2 \zeta \delta (\sqrt{-\gamma}) x^\mu_a x^\nu_b \]  

As we have seen before in (92): \( \delta (\sqrt{-\gamma}) = -\frac{1}{2} \sqrt{-\gamma} \gamma_{ab} \delta \gamma^{ab} \) and \( \delta g^{\mu\nu} = x^{\mu}_a x^{\nu}_b \delta \gamma^{ab} \). In order to derive the energy-momentum tensor the variation of the action should be set to \(-\sqrt{-g} T_{\mu\nu}\) in agreements with (97):

\[ \frac{\delta S}{\delta g^{\mu\nu}} = \mu \int d^2 \zeta \frac{1}{2} \sqrt{-\gamma} \gamma^{ab} \frac{\delta \gamma_{ab}}{\delta g^{\mu\nu}} x^\mu_a x^\nu_b = -\sqrt{-g} T_{\mu\nu} x^\mu_a x^\nu_b \]  

This can be rewritten as:

\[ T^{\mu\nu} \sqrt{-g} = \mu \int d^2 \zeta \sqrt{-\gamma} \gamma^{ab} x^\mu_a x^\nu_b \delta^4 (x^\sigma - x^\sigma (\zeta^a)) \]  

For a straight string in the z-direction in flat space-time directed in the z-direction, the equation above reduces to:

\[ T^{\mu\nu} = \mu \delta(x) \delta(y) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  

because all derivatives are equal to one in flat space-time, and is, as was anticipated, equal to (66). Only the delta-function for the x- and y-components are left, due to the fact that we integrated over the t- and z-components. This is exactly the energy-momentum tensor we have been looking for.

### 4.3 Gravitational Lensing

The Gauss-Bonnet theorem connects the geometry to the topology of a surface [15]. This theory is very useful in our case: knowing the (Gaussian) curvature of the surface and the Euler Characteristic we can derive the deficit angle. Suppose \( \Sigma \)
is a two-dimensional Riemannian manifold, with boundary \( \partial \Sigma \), then the theorem is given by [1]:

\[
\int_{\Sigma} K \cdot dS = 2\pi \chi(M) - \int_{\partial \Sigma} k_g \cdot dl
\]  

(111)

where \( K \) is the Gaussian curvature, in our case given by the Ricci tensor as \( K = R^x_x \). The element \( dS \) is the surface area, and \( dl \) the line element along the boundary of \( \Sigma \), given by: \( dl = rd\theta \). The geodesic curvature is denoted by \( k_g \), and is given by \( \frac{1}{r} \) for a circle. The Euler characteristic (EC) is given by \( \chi(M) \). A complete proof of this theorem can be found in [15]. The Euler characteristic of a surface is given by:

\[
\chi(M) = F - E + V
\]  

(112)

where \( F \) are the faces, \( E \) are the edges and \( V \) are the vertices (corners) of a triangulation of the surface. The Euler characteristics of a disc in two dimensions has \( F = 1, E = 1, V = 4 \), giving \( \chi(M) = 1 \). Inserting the Euler characteristics and the geodesic curvature:

\[
\int R^x_x dS = 2\pi - \int_{\partial \Sigma} \frac{1}{r} rd\theta
\]  

(113)

The second term on the right hand side will give: \( \int_{\partial \Sigma} \frac{1}{r} rd\theta = 2\pi C_r \), where \( A \) is a constant in the function \( C(r) = Ar \). An alternative formulation of the Einstein equation emerges when the Einstein equations are contracted with \( g_{\mu\nu} \):

\[
g^{\mu\nu} R_{\mu\nu} - g^{\mu\nu} g_{\mu\nu} \frac{1}{2} R = g^{\mu\nu} 8\pi G T_{\mu\nu}
\]  

\[ R - 2\mathcal{R} = 8\pi G T^\rho_\rho \]  

(114)\hspace{1cm}(115)

giving \( R = -8\pi G T^\rho_\rho \). Inserting this into the original Einstein equation (64), it is found that:

\[
R_{\mu\nu} = 8\pi G T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = 8\pi G(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\rho_\rho)
\]  

(116)

where \( T^\rho_\rho \) equals the trace of the energy-momentum tensor. For our two-dimensional surface we only need to know the x-components of the Riemann tensor. The x-components of the energy-momentum tensor does not contribute, since the tensor is aligned along the z-axis. The trace of the energy-momentum tensor gives: \( T = 2\mu \delta(x) \delta(y)\mu \). So \( R^x_x \) is given by:

\[
R^x_x = 8\pi G(T^x_x - \frac{1}{2} T) = -8\pi G \delta(x) \delta(y)\mu
\]  

(117)

The Gauss-Bonnet equation (113) becomes:

\[
\int 8\pi G \mu \delta(x) \delta(y)\mu \ dxdy = 8\pi G \mu = 2\pi - 2\pi A
\]  

(118)
it follows that: $A = 1 - 4G\mu$. The cosmic string metric becomes:

$$ds^2 = -dt^2 + dr^2 + dz^2 + (1 - 4G\mu)r^2d\theta^2$$

(119)

An angular wedge of $\delta = 8\pi G\mu$ is removed from flat space-time (fig. 9). When looking at (fig. 10), it can be seen that this structure will cause gravitational lensing. When the angular wedge $\delta$ is removed it will appear that light (the dashed line) travelling from a distant light source, crossing a cosmic string, to an observer, it will appear to the observer that there are two light sources in the sky. This can be visualized as a circle from which a piece has been removed and the sides have been glued together to create a cone. To calculate the lensing angle, the distance of $b$ needs to be calculated. According to geometry, we know:

$$\tan \delta' = \frac{b}{y} = \tan \frac{1}{2}8\pi G\mu$$

(120)

where $\delta' = \frac{1}{2}\delta$. It follows that $b = y \tan \frac{1}{2}8\pi G\mu = y4\pi G\mu$ for $G\mu \ll 1$. The angle $\alpha$ is given by:

$$\tan \alpha = \frac{y4\pi G\mu}{x + y}$$

(121)

Figure 9: Gravitational field of a cosmic string [16].
This gives for the lensing angle $2\alpha$:

$$2\alpha = 8\pi G \mu \left(1 + \frac{x}{y}\right)^{-1}$$  \hspace{1cm} (122)

Looking at fig. 10, it is also understood why a cosmic string can cause density perturbations that eventually cause galaxy formation. A string moving through a region of dust will leave an over-dense accretion region behind it [1]. Consider a string moving through a dusty region (see fig. 11). Because there is an area cut out of space-time caused by the string the dust particles will eventually collide, resulting in an accretion disk in the cosmic string wake. This could give a contribution to primordial density fluctuations [6].
5 Cosmic String Dynamics

In this chapter we will first solve the wave equation of a cosmic string, and use a rewritten constraint to the wave equation to derive a cusp. A cusp is a certain kind of singular string configuration that can form when strings form a loop or can be caused by the gravity of a black hole. Finally some comments will be made on the properties of a cosmic string network in the last section.

5.1 Solution of Cosmic String Wave Equation

Having just derived the effective action of a string (104) we can now derive the equations of motion of the string. A two-dimensional surface is represented by: \( x^\mu = x^\mu(\zeta^a) \), and the line element is defined according to (106) as [6]:

\[
ds^2 = g_{\mu\nu} \frac{\partial x^\mu}{\partial \zeta^a} \frac{\partial x^\nu}{\partial \zeta^b} d\zeta^a d\zeta^b = \gamma_{ab} d\zeta^a d\zeta^b	ag{123}\]

Variation of the effective cosmic string action with respect to \( x^\mu \) gives:

\[
\frac{\delta S}{\delta x^\mu(\zeta^a)} = 0 \tag{124}\]

For arbitrary coordinates, according to (53) the equations of motion become:

\[
\frac{\partial^2 x^\mu}{\partial \zeta^2} + \Gamma^\mu_{\nu\sigma} \gamma^{ab} \frac{\partial x^{\nu}}{\partial \zeta^a} \frac{\partial x^{\sigma}}{\partial \zeta^b} = 0 \tag{125}\]

In Minkowski space (or flat spacetime) \( g_{\mu\nu} = \eta_{\mu\nu} \) and we may set \( \Gamma^\mu_{\nu\sigma} = 0 \), since there is no curvature in flat space time. The equations of motions of a string take the form

\[
\frac{\partial^2 x^\mu}{\partial \zeta^2} = \frac{1}{\sqrt{-\gamma}} \partial_a (\sqrt{-\gamma} \gamma^{ab} \frac{\partial x^\mu}{\partial \zeta^b}) = 0 \tag{126}\]

this is equal to

\[
\partial_a (\sqrt{-\gamma} \gamma^{ab} \frac{\partial x^\mu}{\partial \zeta^b}) = 0 \tag{127}\]

The parameterization of the world-sheet can be chosen freely, for example in the light-cone gauge:

\[
\zeta^0 = t, \quad \zeta^1 = z - t \tag{128}\]

A convenient choice of the gauge of a string in flat space-time is [6]:

\[
\gamma_{01} = 0; \tag{129}\]
\[
\gamma_{00} + \gamma_{11} = 0 \tag{130}\]
with this information we can rewrite the metric as:

\[\gamma_{01} = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial \zeta^0} \frac{\partial x^\nu}{\partial \zeta^1} = \dot{x} \cdot \dot{x}' = 0; \quad (131)\]

\[\gamma_{00} + \gamma_{11} = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial \zeta^0} \frac{\partial x^\nu}{\partial \zeta^0} + \eta_{\mu\nu} \frac{\partial x^\mu}{\partial \zeta^1} \frac{\partial x^\nu}{\partial \zeta^1} = \dot{x}^2 + \dot{x}'^2 = 0 \quad (132)\]

The dots and primes are derivatives with respect to \(\zeta^0 = t\) and \(\zeta^1 = z - t\) respectively. The metric has become conformally flat:

\[\gamma_{ab} = \sqrt{-\gamma} \eta_{ab} \quad (133)\]

\[\gamma^{ab} = \frac{1}{\sqrt{-\gamma}} \eta^{ab} \quad (134)\]

The equation of motion of the string becomes;

\[\partial_a (\sqrt{-\gamma} \gamma^{ab} \frac{\partial x^\mu}{\partial \zeta^b}) = \sqrt{-\gamma} \frac{1}{\sqrt{-\gamma}} \eta^{ab} \partial_a (\frac{\partial x^\mu}{\partial \zeta^b}) = \ddot{x}^\mu - x'^{\mu} = 0 \quad (135)\]

When \(\zeta^0\) is set equal to \(t\), \(\zeta^0 = x^0 = t\), and \(\zeta^1 = \zeta\), the string trajectory can be written as a as a three-vector

\[x(\zeta, t) \quad (136)\]

The string equation of motion becomes the two-dimensional wave equation:

\[\ddot{x} - x'' = 0 \quad (137)\]

where \(x\) depends on a space-coordinate \(\zeta\) and a time-component \(t\). The solution of this two dimensional wave equation is known as d’Alembert’s solution [17] and is derived in appendix C. A general solution of the equation of motion is given by:

\[x(\zeta, t) = \frac{1}{2} [a(\zeta - t) + b(\zeta + t)] \quad (138)\]

representing a left and a right moving wave. The constraints (131) and (132) can be summarized as:

\[\dot{x} \cdot \dot{x}' = 0; \quad (139)\]

\[\dot{x}^2 + \dot{x}'^2 = 1; \quad (140)\]

\[\ddot{x} - x'' = 0 \quad (141)\]

With (139) and (140) and the solution of the wave-equation (138), we can summarize the constraint in a stronger statement. We transform the variables on which \(a\) and \(b\) are depending back to:

\[u = \zeta - t, \quad v = \zeta + t \quad (142)\]

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Now we are going to reformulate the differentiation of \(a\) and \(b\) with respect to \(\zeta\) and \(t\). The reason for this reformulation will become clear during the derivation of the constraint. The differentiation with respect to \(\zeta\) and \(t\) in terms of the new coordinates reads:

\[
a' = \frac{\partial a}{\partial \zeta} = \frac{\partial a}{\partial u} \frac{\partial u}{\partial \zeta} = \frac{\partial a}{\partial u}; \quad (143)
\]

\[
\dot{a} = \frac{\partial a}{\partial t} = \frac{\partial a}{\partial u} \frac{\partial u}{\partial t} = -\frac{\partial a}{\partial u}; \quad (144)
\]

Where we have used (209) and (212). We can conclude that:

\[
a' = -\dot{a} \quad (145)
\]

Along the same line of reasoning, we can take both derivatives of \(b\):

\[
b' = \frac{\partial b}{\partial \zeta} = \frac{\partial b}{\partial v} \frac{\partial v}{\partial \zeta} = \frac{\partial b}{\partial v}; \quad (146)
\]

\[
\dot{b} = \frac{\partial b}{\partial t} = \frac{\partial b}{\partial v} \frac{\partial v}{\partial t} = \frac{\partial b}{\partial v}; \quad (147)
\]

Now we can conclude that:

\[
b' = \dot{b} \quad (148)
\]

With this knowledge we can write the first constraint as:

\[
\dot{x} \cdot x' = \frac{1}{4}[\dot{a} + \dot{b}][a' + b'] = \frac{1}{4}[-a' + b'][a' + b']
\]

\[
= \frac{1}{4}[-a'^2 - ab' + a'b' + b'^2] = 0 \quad (149)
\]

From this equation it follows that: \(-a'^2 + b'^2 = 0\), or:

\[
a'^2 = b'^2 \quad (150)
\]

When we insert the solution (221) in the second constraint we find together with (150) that:

\[
\dot{x}^2 + x'^2 = \frac{1}{4}[(\dot{a} + \dot{b})^2 + (a' + b')^2] = \frac{1}{4}[-a' + b']^2 + \frac{1}{4}[a' + b']^2
\]

\[
= \frac{1}{4}[a'^2 + b'^2 - 2ab' + a'^2 + 2ab' + b'^2]
\]

\[
= \frac{1}{4}[2a'^2 + 2b'^2] = \frac{1}{4}[4a'^2] = 1 \quad (151)
\]

We can conclude that the constraints (139) and (140) can be reformulated as:

\[
a'^2 = 1 = b'^2 \quad (152)
\]
The second constraint can be rewritten as:

\[ \dot{x}^2 = 1 - x'^2 = 1 - \frac{1}{4}[a'^2 + 2a'b' + b'^2] \]

\[ = 1 - \frac{1}{4}a'^2 - \frac{1}{2}a'b' - \frac{1}{4}b'^2 \quad (153) \]

When we use \( a'^2 = b'^2 = 1 \), (153) becomes:

\[ \dot{x}^2 = 1 - \frac{1}{4} - \frac{1}{2}a'b' - \frac{1}{4} = \frac{1}{2} - \frac{1}{2}a'b' = \frac{1}{4} - \frac{1}{2}a'b' + \frac{1}{4} = \frac{1}{4}a'^2 - \frac{1}{2}a'b' + \frac{1}{4}b'^2 \]

\[ = \frac{1}{4}[a'^2 - 2a'b' + b'^2] = \frac{1}{4}[a' - b']^2 \quad (154) \]

This result (\( \dot{x}^2 = \frac{1}{4}[a' - b']^2 \)) will be a useful identity in the next paragraph. Summarizing the results above, the wave equation of a moving string is equal to:

\[ \ddot{x} - x'' = 0 \quad (155) \]

The solution of the wave equation of an oscillating string is equal to:

\[ x(\zeta, t) = \frac{1}{2}\left[a(\zeta - t) + b(\zeta + t)\right] \quad (156) \]

representing a right and a left moving wave. From the two-dimensional metric two constraints can be derived:

\[ \dot{x} \cdot x' = 0 \quad (157) \]

\[ \dot{x}^2 + x'^2 = 1 \quad (158) \]

and be rewritten into: \( a'^2 = b'^2 = 1 \).

### 5.2 Oscillating loops

The motion of a closed loop is described by (156) and (152) with \( \zeta \) varying in the range [6]:

\[ 0 \leq \zeta \leq L \quad (159) \]

where \( L = \epsilon/\mu \) is the invariant length of the loop, \( \mu \) the tension of the string and \( \epsilon \) the total energy, since tension is defined as an energy density (energy per unit of length). For a closed loop, we want the solution to be periodic:

\[ x(\zeta + L, t) = x(\zeta, t) \quad (160) \]

In terms of \( a \) and \( b \) this is written as:

\[ b(\zeta + t + L) - b(\zeta + t) = -a(\zeta - t + L) + a(\zeta - t) = 0 \quad (161) \]
It is also demanded that the function \( a \) and \( b \) are to be periodic:

\[
\begin{align*}
    a(\zeta + L) &= a(\zeta) \quad (162) \\
    b(\zeta + L) &= b(\zeta) \quad (163)
\end{align*}
\]

The motion of the loop must also be periodic in time:

\[
\begin{align*}
    x(\zeta + \frac{L}{2}, t + \frac{L}{2}) &= \frac{1}{2}[a(\zeta + \frac{L}{2} - t - \frac{L}{2}) + b(\zeta + \frac{L}{2} + t + \frac{L}{2})] \\
    &= \frac{1}{2}[a(\zeta - t) + b(\zeta + t + L)] = x(\zeta, t) \quad (164)
\end{align*}
\]

where we used that the function \( b \) is periodic in \( L \). The period is \( \frac{L}{2} \). Because the timescale of the oscillation is comparable to the loop length \( L \), the loop motion must be relativistic. An interesting property of these solutions is that the string can reach the velocity of light during each period. To describe the behaviour of the string near luminal motion (i.e. moving with the speed of light), we choose the space-time coordinates so that the luminal point is at \( \zeta = t = 0 \) and \( x = 0 \). For a continuously differentiable function we can apply a Taylor expansion:

\[
f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + ... \quad (165)
\]

Expanding the functions \( a \) and \( b \) near \( \zeta = 0 \):

\[
\begin{align*}
    a(\zeta) &= a'_0 \zeta + \frac{1}{2!}a''_0 \zeta^2 + \frac{1}{3!}a'''_0 \zeta^3 + ...; \quad (166) \\
    b(\zeta) &= b'_0 \zeta + \frac{1}{2!}b''_0 \zeta^2 + \frac{1}{3!}b'''_0 \zeta^3 + ... \quad (167)
\end{align*}
\]

Where the abbreviation \( a'(\zeta = 0) = a'_0 \) has been used. When we use the constraint \( a'^2 = b'^2 = 1 \) we can write

\[
\begin{align*}
    a'^2 &= (a'_0 + a''_0 \zeta + \frac{1}{2}a'''_0 \zeta^2 + ...)^2; \quad (168) \\
    b'^2 &= (b'_0 + b''_0 \zeta + \frac{1}{2}b'''_0 \zeta^2 + ...)^2 \quad (169)
\end{align*}
\]

these can only be equal if:

\[
\begin{align*}
    a'_0 &= -b'_0; \quad (170) \\
    |a'_0| &= |b'_0| = 1; \quad (171) \\
    a'_0 \cdot a''_0 &= b'_0 \cdot b''_0 = 0; \quad (172) \\
    a''_0 + a'_0 \cdot a''_0 &= b''_0 + b'_0 \cdot b''_0 = 0, ... \quad (173)
\end{align*}
\]
From the equation (154) above we find that:

\[
\dot{x}^2(\zeta, t) = \frac{1}{4} (a' - b')^2 = \frac{1}{4} \left( \left( a_0' + \frac{1}{2!} a_0'' \zeta^2 + \ldots \right) - \left( b_0' + \frac{1}{2!} b_0'' \zeta^2 + \ldots \right) \right)^2
\]

\[
= \frac{1}{4} (a_0' - b_0' a_0'' + b_0'' - b_0'' a_0' + b_0') - \frac{1}{4} (a_0'' \zeta + b_0'' \zeta)^2 + \ldots
\]

\[
= 1 - \frac{1}{4} [((\zeta - t) a_0'' + (\zeta + t) b_0'')^2 + \ldots]
\]

(174)

At the second line (170) - (173) have been used. There exists a point a long this curve for which the speed of the string is equal to the speed of light: \(|\dot{x}| = 1\).

The shape of the string at \(t = 0\), with the use of (166), (167) and (170), is given by:

\[
x(\zeta, 0) = \frac{1}{2} [a + b]
\]

\[
= \frac{1}{2} \left( \left( a_0' + \frac{1}{2} a_0'' \zeta^2 + \frac{1}{3!} a_0''' \zeta^3 + \ldots \right) + \left( b_0' \zeta + \frac{1}{2} b_0'' \zeta^2 + \frac{1}{3!} b_0''' \zeta^3 + \ldots \right) \right)
\]

\[
= \frac{1}{2} (a_0' + b_0') \zeta + \frac{1}{2} \left( \frac{1}{2} a_0'' + \frac{1}{2} b_0'' \right) \zeta^2 + \frac{1}{2} \left( \frac{1}{3!} a_0''' + \frac{1}{3!} b_0''' \right) \zeta^3 + \ldots
\]

\[
= \frac{1}{4} (a_0'' + b_0'') \zeta^2 + \frac{1}{12} (a_0''' + b_0''') \zeta^3 + \ldots
\]

(176)

For \((a_0'' + b_0'') \neq 0\) the string develops a 'cusp', i.e. a certain kind of singularity, at the origin: \(\zeta = t = 0\). The shape of the cusp at the origin is visualised in fig. 12.

The velocity at the cusp approaches the speed of light, as we have seen in (175). With a tension of the string not so far below Planck scale it emits an intense beam of gravitational waves in the direction of its motion [6].

When the string reaches the speed of light, the cusps can emit gravitational waves. There are two specific case when gravitational waves are emitted by a cosmic string, both cases could be detected on earth, but are relatively difficult to discover:

1. Cusps: The cusps beam the radiation in a specific direction, perpendicular to the formed cusp. These gravitational beams have a spectrum that can be distinguished from gravitational waves from other sources. But due to their specifically aimed direction and the number of sources, the chances are slim that they are aimed at the earth, and therefore difficult to detect.
2. Kinks: are formed when contracting loops form a cusp. Just like the ’open’ string it will emit a specific spectrum of gravitational radiation, but are aimed at various directions. Loops are created during intersection of strings and will be discussed in more detail in the next section. The amplitude of the emitted waves is directly connected to the tension of the string. Due to the directionless emission and decrease of the tension, the amplitude of the gravitation waves will decrease fast, making them also difficult to detect.

5.3 String interaction and network evolution

A string network is web of strings that during the expansion of our universe have been intersecting with each other. It turns out that the behaviour of a string network is more involved than one would naively expect. First we will describe the most naive expectation. The Friedmann equations can be used to determine the scaling of the energy density of the cosmic string network. The Friedmann equations assume that the universe is uniform and isotropic. The metric is of the form:

\[ g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^2(t) \end{pmatrix} \]  

(177)

When we consider an ideal fluid, this can be described by [12]:

\[ T^\mu_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix} \]  

(178)
where \( \rho \) is the density of the fluid and \( P \) is the pressure of the fluid. The conservation of the energy-momentum tensor in an expanding universe is equal to the covariant derivative of the energy-momentum tensor being zero:

\[
T^\mu_{\nu\mu} = \frac{\partial T^\mu_\nu}{\partial x^\mu} + \Gamma^\mu_{\alpha\mu}T^\alpha_\nu - \Gamma^\alpha_{\nu\mu}T^\mu_\alpha = 0
\]  

(179)

The conservation law for the expanding universe is equal to:

\[
\frac{\partial \rho}{\partial t} + \frac{\dot{a}}{a}[3\rho + 3P] = 0
\]  

(180)

is known as the continuity equation. The equation of state is given by:

\[
\rho = \omega P
\]  

(181)

When we insert the equation of state into (180)

\[
\frac{\partial \rho}{\partial t} + \frac{\dot{a}}{a}[3\rho + 3\omega \rho] = 0
\]  

(182)

\[
\frac{\partial \rho}{\partial t} + \frac{\dot{a}}{a}3\rho[1 + \omega] = 0
\]  

(183)

\[
\frac{da}{dt} \frac{d\rho}{da} + \frac{\dot{a}}{a}3\rho[1 + \omega] = 0
\]  

(184)

this can be written as:

\[
\frac{\dot{a}}{a} \left[ \frac{d\rho}{d\ln a} + 3\rho(1 + \omega) \right] = 0
\]  

(185)

and

\[
\frac{d\rho}{d\ln a} = -3(1 + \omega)\rho
\]  

(186)

\[
\frac{d\ln \rho}{d\ln a} = -3(1 + \omega)
\]  

(187)

This results in:

\[
\ln \rho = -3(1 + \omega)\ln a + \text{constant}
\]  

(188)

When we exponentiate both sides, we find the following solution:

\[
\rho = \rho_0 a^{-3(1+\omega)}
\]  

(189)

Inserting this equation into the second Friedmann equation we get:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho_0 a^{-3(1+\omega)}
\]  

(190)
where we indicate: \( \tilde{\rho}_0 = \frac{3\pi G}{3} \rho \). To get a solution of \( a \) in terms of the variable \( t \), we try the following ansatz:

\[
    a \propto t^\alpha
\]

The scale factor is now equal to:

\[
    \frac{t^{\alpha-1/2}}{t^\alpha} = t^{-2} = \tilde{\rho}_0 t^{-\alpha(1+\omega)}
\]

Then \( \alpha \) equals \( \frac{2}{3(\omega+1)} \). And this results in the following solution of the equation of state:

\[
    a(t) = a_0 t^{\frac{2}{3(\omega+1)}}
\]

where \( a_0 = \frac{1}{\tilde{\rho}_0} \).

The equation of state is different for every type of matter. The equation of state for respectively radiation, matter and string equals:

\[
    \omega = \frac{1}{3}, \quad P_r = \frac{1}{3} \rho_r
\]

\[
    \omega = 0, \quad P_m = 0, \rho_m = 0
\]

\[
    \omega = -\frac{1}{3}, \quad P_s = -\frac{1}{3} \rho_s
\]

Where the equation of state for strings can be obtained from (110) and (178). When we plug these equations into (189), we find that the density for these type of matter scale respectively as:

\[
    \rho_r \propto a^{-3(1+1/3)} = \frac{1}{a^4(t)}; \quad (197)
\]

\[
    \rho_m \propto a^{-3(1+0)} = \frac{1}{a^3(t)}; \quad (198)
\]

\[
    \rho_s \propto a^{-3(1-1/3)} = \frac{1}{a^2(t)} \quad (199)
\]

The density of cosmic string networks scales as \( \frac{1}{a^2(t)} \) [18], [19].

But as it turns out to be, matter and radiation give the most important contribution, and therefore it is not possible that the strings scale as \( \frac{1}{a^2} \). Cosmic strings are not a large contributor to today’s spectrum, indicating that their contribution is diminished by some events that can not be accounted for in the expression above. There are some events that have not been incorporated in this rather naive calculation. Strings can for example intersect with each other or with themselves, thereby creating loops that gradually decay because the loops emit gravitational waves. So there are two loop formation mechanisms that eventually reduce the string energy density [18]:
1. Collision of two strings: two strings intercommute with each other. During this intersection a loop is created that gradually decays by the means of gravitational radiation.

2. Self-intersection: a single string intersects with itself creating a loop.\footnote{When two strings intersect they can also entangle or simply cross each other without forming a loop, it is not always the case that loops are created during intersection.}

Besides those two loop formation mechanisms, there is also the phenomenon of wiggly strings that has to be considered when looking at options that can reduce the string density. The density of wiggly strings, cosmic strings with kinks or cusps, grow more slowly compared to strings that are stretched [1]. In the first two cases metioned above, the loops begin to oscillate and decay by emitting gravitational waves.

Self-intersection and the collision of two strings is visualised in figure 14. Effectively, a cosmic string network will therefore not scale as found in the (naive)
calculation above, but will scale with a density that does not dominate over matter or radiation. Simulations have shown that effectively the density of strings scales as

$$\rho_s \propto \frac{1}{a^4(t)}$$  \hspace{1cm} (200)

during the radiation dominated area [1]. When the density is now multiplied by the four-volume it will remain constant, thus scale invariant, as can be seen in fig. 13. This means that the cosmic strings scale just like radiation and do not do dominate over them [1]. The density of the string decreases fast enough to be in agreement with present cosmological data.

Gravitational waves emitted by loops will have a relatively broad spectrum and are beamed into several directions, unlike gravitational waves emitted by cusps. Gravitational waves emitted by loops will therefore be more easy to find. Another reason for this is that cusps might only have been formed during the initial stages of the universe, while loops can be created throughout the lifespan of the universe. It will be more likely to detect gravitational radiation emitted by loops than by cusps.

An interesting remark that can be made of the properties of one-dimensional defects (strings) versus two-dimensional defects (domain walls) and zero-dimensional defects (monopoles), concerns their energy density. Between a network of domain walls, the typical spacing will be of order $t$. The energy density in domain walls will then scale as $\frac{1}{t}$, and would dominate over the matter and radiation densities [1]. This network is therefore excluded from the scenario. Point like defects cannot find each other like strings do. The spacing between them scales as $\frac{1}{a(t)}$ and the density as $\frac{1}{a(t)^3}$, just like other massive matter. Point-like defects would come to dominate the radiation-dominated era and can also be excluded.
6 Conclusion

Cosmic strings are cosmological phenomena that came into being during the initial stages of the development of the universe. Three different types of phase transitions were mentioned that could cause the cosmic strings to arise: GUT transition, electroweak transition and the quark-hadron transition. Cosmic strings are a specific type of topological defects that arises during phase transitions. Topological defects can occur when the field symmetries are broken. This happened when the universe cools down below some critical temperature $T_c$ and the field is forced to choose a vacuum phase.

There are several ways to detect cosmic strings. When a cosmic string is formed during one of the phase transitions, it creates a wedge in space-time. Light coming from a distant object that passes a cosmic string before it is observed on earth, will be deformed by the gravity exerted by the string. The lensing causes two exact similar objects to appear in the sky. There have not yet been identified any objects appearing in the sky that could have been lensed by a cosmic string, but if they were found, they would probably look something like fig 15.

To calculate the angular wedge, some derivations were necessary. First of all a little bit of General Relativity has been used to calculate the Einstein gravity of a string. An approximation of the energy-momentum tensor combined with the Riemann tensor were necessary to apply the Gauss-Bonnet theorem. In this case an angular wedge of $\delta = 8\pi G\mu$ is removed from the metric of flat space-time. This can be imagined as a circle from which an angle $\delta$ is removed and the ends are glued together, creating a cone.

Another way of detecting cosmic strings is by the gravitational waves emitted by the cusps or kinks that originate on the strings. Cusps emit gravitational waves in a very particular direction with a specific spectrum. Due to the the specifically aimed directions of these waves they are difficult to detect on earth. Kinks behave like cusps but are formed on contracting loops. The contraction of the loop causes the amplitude of the emitted beams to decrease fast, making them also hard to find.

Multiple strings together form a string network, that is web of strings that during the expansion of our universe have been intersecting with each other. The Friedmann equations can be used to determine the scaling of the energy density of the cosmic string network. Naively this would scale as $\frac{1}{a(t)^2}$, but that’s ruled out because it would imply strings to dominate the energy budget. There are two loop formation mechanisms that eventually reduce the string energy density beyond that of the initial approximation. The final scaling of the mass density of a string
behaves as $\rho_s = \frac{1}{\sigma(t)}$, and therefore does not dominate over matter or radiation.

Even though no detection of a cosmic string has been reported, there were some observations that were first believed to be explained by cosmic strings. The first possible observation of a cosmic string was done by Sazhin. On January 12, 2006 the Hubble Space telescope observed his double extragalactic object [20]. They reported the observation of a lensing candidate called CSL-1 (Capodimonte-Sternberg Lens Candidate number one), see fig. 15. There are also three other candidates, going by the not so startling names CSL-2 through CSL-4. The two images of CSL-1 separated by 2” look almost identical, both have the same redshift and magnitudes. If the images were indeed of the same galaxy, they had to have been created due to a cosmic lens, as we have just seen in paragraph 4.3 [19].

![Figure 15: Image of the region surrounding CSL-1 [20].](image)

Unfortunately, the high-resolution image of CSL-1 showed that the object is in fact a pair of giant interacting elliptical galaxies [20]. Despite of the similarities in energy and light distribution and radial velocities, CSL-1 is not the lensing of an cosmic string. Another argument supporting that CSL-1 is not created by a cosmic lens, is that all objects falling inside the narrow strip defined by the deficit angle computed along the string should be affected, and more lensed images are to be expected. Examination of the area around CSL-1 revealed no more lensed objects, leading to the rejection of the cosmic string hypothesis.

Except for observing of the effect of lensing there is a different way of detecting cosmic strings. A recent article of Bevis and Hindmarsh et al. [21] compared models of the cosmic microwave background (CMB) power spectra with CMB models
including cosmic strings. The CMB data give a moderate preference to the model including cosmic strings. The widely accepted inflation model also fits the CMB data. Including cosmic strings in this scenario might better explain the sourcing of additional anisotropies in the CMB radiation. In the combined inflation plus string scenario, inflation creates primordial perturbations still visible today, and the cosmic strings cause additional perturbations. The observed CMB anisotropies are considered to be small, therefore the coupling between inflation and cosmic strings perturbations are ignored. The string and inflation perturbations are calculated separately and are simply added together in order to give the total power spectrum. So including the cosmic string in the inflation model gives better results when compared with the actual CMB data.

To summarize: cosmic strings are an interesting topic to study from a theoretical point of view. Finding cosmic strings would clearly have serious implications for physics and allowing for a good chance to distinguish between different candidates for a fundamental theory [22].
A Higgs boson

According to Goldstone’s theorem, models showing spontaneous symmetry breaking lead to massless scalar particles. According to the standard model, at temperatures above a critical temperature, when symmetry is unbroken and all elementary particles are massless, with the exception of the Higgs particle. A popular statement about the Higgs boson is that it gives mass to massless particles, in particular the particles described by the electro-weak nuclear force. Below this critical temperature, the Higgs field spontaneously breaks into a vacuum expectation value for the Higgs field. To explain the mechanism of the Higgs-boson we need a slightly different version of our previously encountered Abelian-Higgs model. To explain the Higgs mechanism, we start with the following Lagrangian:

$$\mathcal{L} = D_\mu \phi D^\mu \phi^* - V(\phi, \phi^*) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$  \hspace{1cm} (201)

where $\phi$ is a complex scalar field and the anti-symmetric tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The partial derivative is now replaced by the covariant derivative and is given by: $D_\mu = \partial_\mu - ieA_\mu$, with $e$ the gauge coupling and $A_\mu$ the gauge vector field and $V(\phi, \phi^*)$ our familiar Mexican hat potential (5). This model is invariant under the following transformations:

$$\phi(x) \rightarrow e^{i\alpha(x)} \phi(x); \hspace{1cm} (202)$$

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x) \hspace{1cm} (203)$$

This symmetry is spontaneously broken and the Higgs field acquires an expectation value not equal to zero. To study the properties of the different particles, represent $\phi$ as $\phi = \eta + \phi_1$. The Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_1)^2 - \frac{1}{2} \lambda \eta^2 |\phi_1|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e^2 \eta^2 A_\mu A^\mu + \mathcal{L}_{int} \hspace{1cm} (204)$$

where $\mathcal{L}_{int}$ includes the second and higher order terms in $\phi_1$ and $A_\mu$. This Lagrangian also includes a massive scalar particle: the Higgs-boson with mass $m_H = \sqrt{\lambda} \eta$. As of yet, the Higgs boson has not been found.
Derrick’s theorem

Derrick’s theorem states that there are no stable time-independent, localized solutions for any scalar models in more than one dimension. This theorem uses an scaling argument proposing their non-existence. A brief outline of the proof is sketched below.

Consider the energy of a localized solution $\phi(x)$ in n-dimensions [6], [23]:

$$E = \int d^n x \left[ (\nabla \phi)^2 + V(\phi) \right] = I_1 + I_2$$  \hspace{1cm} (205)

where $I_1$ represents the gradient term and $I_2$ for the potential term. Suppose we rescale $\phi(x)$ by: $x \rightarrow \alpha x$. The rescaled energy becomes:

$$E_\alpha = \alpha^{2-n} \int d^n x (\nabla \phi)^2 + \alpha^{-n} \int d^n x V(\phi) = \alpha^{2-n} I_2 + \alpha^{-n} I_2$$  \hspace{1cm} (206)

When $n \geq 2$ the equation above will collapse. Fortunately, there are some ways in which the unfortunate outcome of Derrick’s theorem can be avoided. This can either be done by adding higher derivative terms, or by allowing time-dependency.
C String dynamics

The solution to the two-dimensional wave-equation:

$$\ddot{x} - x'' = 0$$  \hspace{1cm} (207)

First two new variables are introduced, let’s call them $u$ and $v$:

$$u = \zeta + t, \quad v = \zeta - t$$  \hspace{1cm} (208)

The three-vector now depends on: $x(\zeta, t) \rightarrow x(u, v)$. When $u$ and $v$ are differentiated with respect to $\zeta$ then:

$$\frac{\partial u}{\partial \zeta} = 1, \quad \frac{\partial v}{\partial \zeta} = 1$$  \hspace{1cm} (209)

When differentiating $x(u, v)$ once with respect to $\zeta$:

$$\frac{\partial x}{\partial \zeta} = \frac{\partial x}{\partial u} \frac{\partial u}{\partial \zeta} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial \zeta} = \frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}$$  \hspace{1cm} (210)

At the second equality (209) has been used. When differentiating again, the right hand side of (137) becomes:

$$\frac{\partial^2 x}{\partial \zeta^2} = \frac{\partial}{\partial \zeta} \left( \frac{\partial x}{\partial u} - \frac{\partial x}{\partial v} \right) = \frac{\partial}{\partial u} \left( \frac{\partial x}{\partial u} - \frac{\partial x}{\partial v} \right) + \frac{\partial}{\partial v} \left( \frac{\partial x}{\partial u} - \frac{\partial x}{\partial v} \right) \frac{\partial v}{\partial \zeta}

= \frac{\partial^2 x}{\partial u^2} - 2 \frac{\partial^2 x}{\partial u \partial v} + \frac{\partial^2 x}{\partial v^2}$$  \hspace{1cm} (211)

This can also be done for the left-hand-side of (137). When we differentiate the new variables with respect to $t$:

$$\frac{\partial u}{\partial t} = 1, \quad \frac{\partial v}{\partial t} = -1$$  \hspace{1cm} (212)

Differentiating $x(u, v)$ once with respect to $t$ gives:

$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial t} = \frac{\partial x}{\partial u} - \frac{\partial x}{\partial v}$$  \hspace{1cm} (213)

At the second equality (212) has been used. When differentiating again, the left hand side of (137) becomes:

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial x}{\partial u} - \frac{\partial x}{\partial v} \right) = \frac{\partial}{\partial u} \left( \frac{\partial x}{\partial u} - \frac{\partial x}{\partial v} \right) \frac{\partial u}{\partial t} + \frac{\partial}{\partial v} \left( \frac{\partial x}{\partial u} - \frac{\partial x}{\partial v} \right) \frac{\partial v}{\partial t}

= \frac{\partial^2 x}{\partial u^2} - 2 \frac{\partial^2 x}{\partial u \partial v} + \frac{\partial^2 x}{\partial v^2}$$  \hspace{1cm} (214)
When we insert (211) and (214) into (137):

\[
\frac{\partial^2 \mathbf{x}}{\partial u^2} - 2 \frac{\partial^2 \mathbf{x}}{\partial u \partial v} + \frac{\partial^2 \mathbf{x}}{\partial v^2} - 2 \frac{\partial^2 \mathbf{x}}{\partial u \partial v} - \frac{\partial^2 \mathbf{x}}{\partial v^2} = 0 \tag{215}
\]

This results in:

\[
4 \frac{\partial^2 \mathbf{x}}{\partial u \partial v} = 0 \tag{216}
\]

The wave equation becomes:

\[
\frac{\partial^2 \mathbf{x}}{\partial v \partial u} = 0 \tag{217}
\]

By integrating this equation twice, the general solution can be derived. When integrating (217) once:

\[
\frac{\partial \mathbf{x}}{\partial u} = c(u) \tag{218}
\]

Performing the integration again gives:

\[
\mathbf{x}(u, v) = \int c(u) \, du + a(v) \tag{219}
\]

The integral is a function of \( u \), let’s assume that the integral is equal to some \( b(u) \), then the solution becomes:

\[
\mathbf{x}(u, v) = b(u) + a(v) = b(\zeta + t) + a(\zeta - t) \tag{220}
\]

A general solution of the equation of motion is given by:

\[
\mathbf{x}(\zeta, t) = \frac{1}{2} [a(\zeta - t) + b(\zeta + t)] \tag{221}
\]

representing a left and a right moving wave.
References

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[2] Figure adapted from: ireswww.in2p3.fr/ires/obernai/greiner/greiner16.pdf


[9] Figure adapted from the Cambridge Cosmology website: www.damtp.cam.ac.uk/user/gr/public.html


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