Dilute Temperley-Lieb algebra

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Abstract

In this thesis the regular and dilute \( n \)-diagram algebra, with the variable \( \beta = q^{-1} + q \) are studied, the regular and dilute Temperley-Lieb algebra are introduced to give a more abstract representation of the corresponding \( n \)-diagram algebra and their equivalence is proven. In the case that \( q \) is not a root of unity a full set of irreducible modules is given and after proving the irreducibility of these modules it is shown that the algebras are semisimple using Wedderburn’s theorem. After that transfer matrices are introduced and shown how they relate with the Temperley-Lieb algebra, these transfer matrices are then used to show the relation between the Potts model on the square lattice and the Temperley-Lieb algebra. Then a \( O(n) \) model on the square lattice is examined and the corresponding phase diagram is explored further.
1. Introduction

Using statistical physics it is possible to describe many systems and physical properties. A very useful tool in the examining many of these is the Temperley-Lieb algebra. Ever since its introduction almost fifty years ago [1] it has been used to help solve many systems in statistical physics.

In this thesis we will be examining the Temperley-Lieb algebra and one of its generalizations, the dilute Temperley-Lieb algebra. There exist two ways to describe these algebras. The first is using diagrams consisting of a vertical strip with $n$ points on the upper boundary and $n$ points on the lower boundary, then a spanning set of the algebra corresponds to all the possible ways of connecting those points by non-intersecting lines. The second is purely algebraically using generators and relations between those generators. For the Temperley-Lieb algebra the equivalence between the two has been proven[5]. But for the dilute Temperley-Lieb algebra such an equivalence has never been proven. One of the main problems with proving such an equivalence is that is hard to determine whether the relations between generators that have been found do in fact completely determine the algebra, because there might be some other relation that have been overlooked.

In this thesis we will first examine the Temperley-Lieb algebra and show the equivalence between the two ways of describing it. Then we will examine the representations of this algebra, similar to [5], by constructing a set of modules and showing that in certain cases these are irreducible and form a complete set of irreducible modules.

Then we will look at the dilute Temperley-Lieb algebra. We will define the way of viewing the algebra using diagrams and give a construction for each diagram using a set of generators. After that we define an algebra in terms of generators and defining relations and show that this algebra is isomorphic to the dilute $n$-diagram algebra, which is the algebra defined by the diagrams, this has not been done before. Then we will study the representations of this algebra, similar to [6], by constructing a set of modules and showing that in certain cases they form a set of complete irreducible modules.

After that we will give a short recap of some elements of statistical physics and then introduce the transfer matrix. An important part of statistical physics is the calculating of partition functions and the transfer matrix is a useful way of rewriting the problem to make it possible to calculate the partition function. We will show the link between the transfer matrix on square lattice models and the Temperley-Lieb algebra.

Then we will examine two square lattice models, the Potts model and the $O(n)$ model. We will show how the transfer matrix can be applied to these two models and finally we will examine the phase diagram of an $O(n)$ model on a square lattice. This lattice model consists of a grid where each edge corresponds to a particle and each vertex to an interaction between the adjacent particles. There are four different types of interaction on each interaction and each of those is given a weight. This weight corresponds to the chance that that particular interaction takes place on any vertex. Different weights lead to different lattices and at certain weight a phase transition takes place. We then try to explore when such a transition might take place.
I would like to thank my supervisors prof. dr. Bernard Nienhuis and prof. dr. Jasper Stokman not just for consistently finding the time to discuss my progress, but also for the many insights they have given me into this fascinating subject. Finally i would like to thank Kayed Al-Qasemi MSc for helping understand the subject matter and teaching me how to draw the various diagrams in this thesis.
2. Temperley-Lieb algebra

We will start this chapter by introducing the $n$-diagram algebra and showing its equivalence with the Temperley-Lieb algebra by examining the dimensions of both algebras. Then we will begin our study on the representations of the $n$-diagram algebra.

The $n$-diagram algebra is defined as follows. We draw two horizontal lines and mark $n$ points on each of the lines. We then connect each of these points with $n$ non-intersecting lines which are fully within the horizontal lines, we identify these diagrams up to isotopy. This will be an $n$-diagram. Next we form the complex vector space with basis the set of all $n$-diagrams. We then define multiplication between two $n$-diagrams by identifying the bottom horizontal line of the right diagram with the top horizontal line of the diagram, replacing each closed loop by a factor $\beta$ and then removing the interior horizontal line. We will from now on refer to this as concatenation. For example:

This defines for each $\beta \in \mathbb{C}$, an associative algebra, the $n$-diagram algebra which we will denote $D_n$. This algebra has a unit namely:

It is often useful to write $\beta = q + q^{-1}$, where $q \in \mathbb{C}^\times$.

**Definition 2.1.** For a $\beta \in \mathbb{C}$, the Temperley-Lieb algebra denoted $TL_n(\beta)$ or $TL_n$ is defined as being generated by the unit $1$ and elements $u_i, i \in \{1, 2, \ldots, n-1\}$, satisfying

\[ u_i^2 = \beta u_i, \quad u_iu_{i+1}u_i = u_i \quad \text{and} \quad u_iu_j = u_ju_i \quad \text{if} \quad |i - j| > 1. \]

For the details of this construction see appendix A.

**Proposition 2.2.** There exists a unique unit-preserving surjective algebra morphism $\phi : TL_n \rightarrow D_n$ such that $\phi$ maps the generators of $TL_n$ to the following diagrams

\[ 1 \mapsto \begin{array}{ccc} & & \\ & & \\ & & \end{array} \quad \text{and} \quad u_i \mapsto \begin{array}{ccc} & & \end{array}. \]
Proof. To prove this we need to check that all the relations in $\text{TL}_n$ also hold in the $n$-diagram algebra with the given identifications,

\[
\begin{align*}
    u_i^2 &= \beta \\
    u_i u_{i-1} u_i &= \text{[diagram]} \\
    u_i u_j &= \text{[diagram]} \\
    &= u_j u_i, \quad |i - j| > 1.
\end{align*}
\]

It can be shown that the identity and the $\phi(u_i)$ generate the $n$-diagram algebra. From this it follows that the $n$-diagram algebra is a quotient of the Temperley-Lieb algebra. To prove that the two are in fact isomorphic we will show that their dimensions are the same. First we will calculate the dimension of the $n$-diagram algebra. For this we will introduce $(n;k)$-link diagrams. We construct these by drawing a single horizontal line with $n$ points. We can then connect these dots with lines as in the regular $n$-diagrams except now $k$ lines will be connected to only one point. We will call these lines defects. We will call a line that is connected to two points a link. Note that by construction $n - k$ and $n + k$ are even. An example for $n = 7$ and $k = 1$

We will always orient the $(n,k)$-link diagrams so that the defects face up.

We can always transform an $n$-diagram into an $(2n,0)$-link state by connecting the two horizontal lines on the left and reorienting the rest of the diagram so that this line forms the bottom of the diagram. For example:
It is obvious that this creates a bijection between the set of $n$-diagrams and the set of $(2n, 0)$-link diagrams.

**Lemma 2.3.** There exists a bijection between the $(n, k)$-link diagrams and the increasing walks on $\mathbb{Z}^2$ from $(0, 0)$ to $\left(\frac{n+k}{2}, \frac{n-k}{2}\right)$, which do not cross the diagonal $\{(m, m) | m \in \mathbb{Z}\}$. Increasing here means that in the walk you can only move up or to the right.

**Proof.** We give this bijection by reading an $(n, k)$-link state from the left to the right and you move up at the $i$-th step if you close a link at the $i$-th point. Otherwise you move to the right. For the earlier example with $n = 7$ and $k = 1$,

There are $\frac{n-k}{2}$ links in the diagram and therefore we take precisely $\frac{n-k}{2}$ steps up and $n - \frac{n-k}{2} = \frac{n+k}{2}$ to the right. We cannot cross the diagonal because to close a link we first have to open it. Which means to take a step up we need to move to the right, so we can’t cross the diagonal. This is a bijection because we can easily reverse this process by reversing the walk, moving from the right to the left in the $(n, k)$-link state diagram by opening a link if we move down, If we move down we close a link if possible otherwise we create a defect. This clearly creates a bijection between the walks and $(n, k)$-link states. \(\square\)

**Lemma 2.4.** There exists a bijection between the Temperley-Lieb algebra and the increasing walks on $\mathbb{Z}^2$ from $(0, 0)$ to $(n, n)$ which do not cross the diagonal $\{(m, m) | m \in \mathbb{Z}\}$.

We will not prove this in this thesis, for the details view [5].

**Theorem 2.5.** The $n$-diagram algebra is isomorphic to the Temperley-Lieb algebra.

**Proof.** We have previously seen that the dimension of the $n$-diagram algebra equal is to the number of $(2n, 0)$-link diagrams and from lemma 2.3 we know that this is equal to the number of increasing walks on $\mathbb{Z}^2$ from $(0, 0)$ to $(n, n)$ which do not cross the diagonal $\{(m, m) | m \in \mathbb{Z}\}$. From lemma 2.4 we know that the same holds true for $TL_n$. Finally we know that the $n$-diagram algebra is a quotient of $TL_n$ which means that the two algebras are in fact isomorphic because they have the same dimension. \(\square\)

**Lemma 2.6.** If $d_{n, k}$ is the dimension of increasing walks on $\mathbb{Z}^2$ from $(0, 0)$ to $\left(\frac{n+k}{2}, \frac{n-k}{2}\right)$ that do not cross the diagonal then,

$$d_{n, k} = \binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k-2}{2}}.$$

**Proof.** Any path ending at $\left(\frac{n+k}{2}, \frac{n-k}{2}\right)$ must pass through $\left(\frac{n+k}{2} - 1, \frac{n-k}{2}\right)$ or $\left(\frac{n+k}{2}, \frac{n-k}{2} - 1\right)$ but not both, so we have the recursion

$$d_{n, k} = d_{n-1, k+1} + d_{n-1, k-1}.$$
With the boundary values \( d_{n,n} = 1 \), because there is only one path from \((0,0)\) to \((n,0)\) and \( d_{n,-1} = 0 \), because the walk cannot cross the diagonal. You can easily check that the solution is,

\[
d_{n,k} = \binom{n-k}{\frac{n}{2}} - \binom{n-1}{\frac{n}{2}}.
\]

**Corollary 2.7.** The dimension of the Temperley-Lieb algebra is,

\[
\dim TL_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}.
\]

**Proof.** The dimension of the Temperley-Lieb algebra is equal to \( d_{2n,0} \), so the rest follows from the previous lemma.

### 2.1. Representation of the \( n \)-diagram algebra

The \((n,k)\)-link states are not only useful for calculating the dimension of the algebra, but they also serve as left modules for the \( n \)-diagram algebra.

**Definition 2.8.** The complex span over the \((n,k)\)-link states (for all \( k \)) is \( M_n \). We will call this the **link module**.

As the name might suggest \( M_n \) naturally serves as a left \( D_n \)-module by concatenation of the diagram with the link-state. For example,

\[
= \quad = \quad = 
\]

**Lemma 2.9.** *This operation gives \( M_n \) the structure of a left \( D_n \)-module.*

This is fairly obvious since concatenation is associative.

Note that this action can decrease the number of defects by closing a pair of defects, but never increase it. So we can define a submodule of \( M_n \) by viewing only those link states with at most \( k \) defects. We will denote this as \( M_{n,(n-k)/2} \). Since the number of defects can only decrease we have the following inclusions

\[
0 \subset M_{n,[n/2]} \subset \cdots \subset M_{n,1} \subset M_{n,0} = M_n.
\]

Note that these submodules give the lower bound \( \frac{n-k}{2} \) to the number of links when the submodule can contain at most \( k \) defects. We will from now on often denote \( l = \frac{n-k}{2} \) to be the number of links in a \((n,k)\)-link state.
Definition 2.10. We define the standard modules $V_{n,l}$ to be the quotient

$$V_{n,l} \simeq M_{n,l}/M_{n,l+1} \quad \text{for} \quad 0 \leq l \leq \lfloor n/2 \rfloor - 1 \quad \text{and} \quad V_{n,\lfloor n/2 \rfloor} = M_{n,\lfloor n/2 \rfloor}.$$ 

We will identify $V_{n,l}$ with the vector space spanned by the $(n,k)$-link diagrams, this is well-defined because every equivalence class contains precisely one $(n,k)$-link diagram. The $V_{n,l}$ are $D_n$-modules with an action consisting of concatenation when the number of defects is conserved and 0 otherwise. The dimension of these is the same as $d_{n,k}$ given by lemma 2.6.

2.2. Irreducibility of the standard modules

Now it is a reasonable thing to wonder whether the standard modules defined in definition 2.10 are irreducible as $D_n$-modules. As it turns out most of the time that is the case, but in order to show that we will need to introduce an invariant bilinear form. By representing this bilinear form using a matrix we will be able to determine whether the standard modules are irreducible for almost all cases.

The $(n,k)$-link states were oriented so that they were facing to the top allowing for a natural left $D_n$ action. We can of course reflect these $(n,k)$-link states across the horizontal line and in doing so we obtain a left $D_n$ action. We will use these reflected link states to define the following bilinear form.

Definition 2.11. The bilinear form $\langle \cdot, \cdot \rangle_{n,l}$ is defined on $V_{n,l}$ as follows. If $x$ and $y$ are two $(n,k)$-link states, $\langle x, y \rangle_{n,l}$ is computed by concatenating the reflected link state of $x$ with $y$. If not every defect in $x$ is connected with a defect in $y$, $\langle x, y \rangle_{n,l} = 0$. Otherwise $\langle x, y \rangle_{n,l} = \beta^m$ where $m$ equals the number of closed loops.

We will now start of by giving a couple of examples in $V_{4,1}$

$$\langle \begin{array}{c} \includegraphics{example1} \end{array}, \begin{array}{c} \includegraphics{example2} \end{array} \rangle_{n,l} = \begin{array}{c} \includegraphics{example3} \end{array} = \beta$$

$$\langle \begin{array}{c} \includegraphics{example4} \end{array}, \begin{array}{c} \includegraphics{example5} \end{array} \rangle_{n,l} = \begin{array}{c} \includegraphics{example6} \end{array} = 1$$

$$\langle \begin{array}{c} \includegraphics{example7} \end{array}, \begin{array}{c} \includegraphics{example8} \end{array} \rangle_{n,l} = \begin{array}{c} \includegraphics{example9} \end{array} = 0$$

This bilinear form is symmetric, $\langle x, y \rangle$ and $\langle y, x \rangle$ are reflections of each other when we view the concatenation of $x$ and $y$, therefore they have the same value.

We will now consider reflection of an $n$-diagram across a horizontal line. This will be another $n$-diagram so doing this will define a linear map from the $n$-diagram algebra to itself.
This preserves the multiplication except that the order will be reversed, so this is an anti-algebra morphism. We will denote the reflection of a diagram $U$ by $U^\dagger$. It is clear from the diagrams that $1^\dagger = 1$ and $u_i^\dagger = u_i$. Using this we can show that the adjoint of the action with respect to the bilinear form is given by this reflection.

**Lemma 2.12.** For all $n$-diagrams $u$ the bilinear form $(\cdot, \cdot)_{n,l}$ on $V_{n,l}$ satisfies

$$(x, U y)_{n,l} = (U^\dagger x, y)_{n,l} \quad \forall x, y \in V_{n,l}$$

The result immediately follows from comparing the diagrams of the two sides.

**Definition 2.13.** We define the bilinear map $[\cdot, \cdot]_{n,k} : V_{n,l} \times V_{n,l} \to D_n$ as follows. For $x$ and $y$ $(n, k)$-link diagrams we let $[x y]$ be the $n$-diagram formed by viewing the link diagram $x$ as the bottom row in an $n$-diagram, the reflected state of the link diagram $y$ as the top row and connecting each defect with one on the opposite row. We then extend this linearly to $V_{n,l} \times V_{n,l}$ to obtain the map.

Note that connecting the defects is only possible in a unique way so this map is well-defined. This definition gives rise to another way of determining the dimension of the $n$–diagram algebra.

**Proposition 2.14.** The dimension of the $n$-diagram algebra is given by

$$\dim D_n = \sum_{l=0}^{[n/2]} (\dim V_{n,l})^2$$

**Proof.** Because for each $n$-diagram $D$ there exists a unique $l$ and unique $x, y \in V_{n,l}$ such that $D = [x y]$. Therefore the dimension of the $n$-diagram algebra is precisely as in the proposition.

We can combine the bilinear map $[\cdot, \cdot]_{n,l}$ with the bilinear form we have previously defined to obtain the following relation.

**Lemma 2.15.** If $x, y, z \in V_{n,l}$, then

$$[x y] z = (y, z)_{n,l} x.$$  

**Proof.** Because of linearity it suffices to prove it for when $x$, $y$ and $z$ are $(n, k)$-link states. We will prove this lemma by looking at two cases, when at least one of the defects in $y$ are closed by a link in $z$ and when none of the defects in $y$ are closed by a link in $z$.

In the first case we have two defects in $y$ which are connected, so by definition $(y, z)_{n,l} = 0$. For $[x y] z$ we know that every defect in $x$ is connected to one in $y$ and at least two of those defects are closed by a link in $z$. This means that $[x y] z$ has more links than $x$, because the rest of the action can only increase the number of links. But $V_{n,l}$ consists all the link states with precisely $l$ links, therefore $[x y] z = 0$.

In the second case it is clear that the left-hand side of the equation is proportional to $x$, since all the defects in $x$ remain defects. By definition this constant is given by $\beta$ to the power of the number of closed loops. But because $x$ does not cause any closed loops in $[x y] z$ this is precisely $(y, z)_{n,l}$.  

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Proposition 2.16. If $(\cdot,\cdot)_{n,l}$ is not identically zero on $V_{n,l}$, then $V_{n,l}$ is cyclic and indecomposable.

Proof. Since $(\cdot,\cdot)_{n,l}$ is non-zero we have $y, z \in V_{n,l}$ such that $(y, z)_{n,l} = 1$, but then for every $x \in V_{n,l}$ we have $|x\bar{y}| \in D_n$ satisfying

$$|x\bar{y}|z = (y, z)_{n,l}x = x.$$ 

Therefore the module $V_{n,l}$ is cyclic.

Suppose now that $V_{n,l} = A \oplus B$ for some non-trivial submodules $A, B$ of $V_{n,l}$. Then we must have $z = a + b$ for some $a \in A$ and $b \in B$. Suppose now that $(y, a)_{n,l} \neq 0$, then for every $x \in V_{n,l}$ we have that $x = |x\bar{y}|a = \frac{x}{(y, a)_{n,l}}\bar{y}a$.

but $\frac{|x\bar{y}|}{(y, a)_{n,l}} \in D_n$, so $A = V_{n,l}$. But then $B = \{0\}$, while we assumed that $B$ was non-trivial. Therefore we have that $(y, a)_{n,l} = 0$. From this it follows that $(y, b)_{n,l} = 1$ but then $b$ generates $V_{n,l}$ which implies $A = \{0\}$. Therefore $V_{n,l}$ is indecomposable.

We note that $(\cdot,\cdot)_{n,l}$ is not identically zero when $l' \neq 0$ because if we choose any link diagram $x \in V_{n,l}$, $(x, x)_{n,l} = \beta^l$.

Proposition 2.17. If $(\cdot,\cdot)_{n,l}$ is not identically zero

$$V_{n,l} \simeq V_{n,l'} \implies l = l'.$$

Proof. Suppose $\theta : V_{n,l} \to V_{n,l'}$ is an isomorphism and $l \neq l'$ we can assume that $l > l'$ because $\theta$ is invertible. Then choose $y, z \in V_{n,l}$ such that $(y, z)_{n,l} = 1$. For all $x \in V_{n,l}$ we have that $|x\bar{y}|(z) = \theta(|x\bar{y}|z) = \theta(x)$.

But $|x\bar{y}|(z) = 0$ because for $\theta(z)$ left-multiplying with $|x\bar{y}|$ leads to at least $l$ links which is zero in $V_{n,l'}$. From this contradiction it follows that $l = l'$.

To determine the irreducibility of $V_{n,l}$ we will now introduce the following.

Definition 2.18. The radical $R_{n,l}$ of the bilinear form on $V_{n,l}$ is the following subset of $V_{n,l}$

$$R_{n,l} = \{ x \in V_{n,l} : (x, y)_{n,l} = 0 \ \forall y \in V_{n,l} \}.$$ 

The usefulness of this radical immediately follows from the following lemma.

Lemma 2.19. The radical $R_{n,l}$ is the maximal submodule of $V_{n,l}$.

Proof. The radical being a submodule is a direct consequence of lemma 2.12. This submodule is maximal because from lemma 2.15 it follows that every element outside of the radical is a generator of $V_{n,l}$.

We will now determine the irreducibility of $V_{n,l}$ based on the bilinear form $(\cdot,\cdot)_{n,l}$. For this we first recall that the the $(n, n-2l)$-link states form a basis for $V_{n,l}$. We can then represent the bilinear form by a symmetric $d_{n,n-2l} \times d_{n,n-2l}$ matrix.
Definition 2.20. The gram matrices denoted $G_{n,l}$ are the matrices representing the bilinear form $\langle \cdot , \cdot \rangle_{n,l}$ in the basis of $(n, n-2l)$-link diagrams.

A few examples for $n = 4$, 

$$G_{4,0} = (1), \quad G_{4,1} = \begin{pmatrix} \beta & 1 & 0 \\ 1 & \beta & 1 \\ 0 & 1 & \beta \end{pmatrix} \quad \text{and} \quad G_{4,2} = \begin{pmatrix} \beta^2 & \beta \\ \beta & \beta^2 \end{pmatrix},$$

when we use the ordered bases

$$\{1, 1, 1, 1\}, \quad \{\begin{array}{c} \circ \circ \circ \\ \circ \circ \circ \end{array}, \begin{array}{c} \circ \circ \circ \\ \circ \circ \circ \end{array}, \begin{array}{c} \circ \circ \circ \\ \circ \circ \circ \end{array}\}$$

and

$$\{\begin{array}{c} \begin{array}{c} \circ \circ \circ \circ \\ \circ \circ \circ \circ \end{array}, \begin{array}{c} \circ \circ \circ \circ \\ \circ \circ \circ \circ \end{array} \end{array}\}.$$

We now note that the radical $R_{n,l}$ precisely corresponds to the kernel of $G_{n,l}$, which means that the irreducibility of $V_{n,l}$ is equivalent to $\det G_{n,l} \neq 0$. We will determine using restriction of the standard modules a recursion relation for $\det G_{n,l}$ for when $q$ is not a root of unity, where we recall that $\beta = q + q^{-1}$.

Proposition 2.21. We consider the inclusion of the $D_{n-1}$ into $D_n$ by letting the diagrams be the same at the first $n-1$ points and having a vertical line at the $n$'th position. We denote the restriction of $V_{n,l}$ to a $D_{n-1}$-module by $V_{n,l}$ then we have the short exact sequence of $D_{n-1}$ modules,

$$0 \to V_{n-1,l} \to V_{n,l} \to V_{n-1,l-1} \to 0.$$

Proof. The inclusion $V_{n-1,l} \to V_{n,l}$ is given by extending every $(n-1, n-2l-1)$-link state by adding a defect at position $n$. This is an injective $D_{n-1}$ homomorphism because that defect at position $n$ will be conserved under the inclusion of $D_{n-1}$ into $D_n$. Now it remains to show that the quotient $V_{n,l} / V_{n-1,l} \cong V_{n-1,l-1}$.

We note that the quotient $V_{n,l} / V_{n-1,l}$ has a basis consisting of all the cosets represented by $(n, n-2l)$-link states where a link ends at position $n$. Then we have the obvious vector isomorphism $\phi$ where we replace that link with just a defect at position where it began which we will denote by $m$. Now it remains to show that for all $u_i \in D_{n-1}$ and $z$ basis elements of $V_{n,l} / V_{n-1,l}$ that $\phi(u_i z) = u_i \phi(z)$. Note that unless $i = m - 1$ and $z$ has a defect at position $m-1$, $u_i$ simply gives another basis element of $V_{n,l} / V_{n-1,l}$ where the link ending at $n$ begins in some $m'$. For this case it is easy to see that $\phi(u_i z) = u_i \phi(z)$. If $i = m - 1$ and $z$ has a defect at $m-1$, then $u_{m-1} z$ has a defect at the $n$'th position.

\[
\begin{array}{c}
\begin{array}{c} \circ \circ \circ \\ \circ \circ \circ \end{array}, \begin{array}{c} \circ \circ \circ \\ \circ \circ \circ \end{array}, \begin{array}{c} \circ \circ \circ \\ \circ \circ \circ \end{array} = \begin{array}{c} \begin{array}{c} \circ \circ \circ \circ \\ \circ \circ \circ \circ \end{array}, \begin{array}{c} \circ \circ \circ \circ \\ \circ \circ \circ \circ \end{array} \end{array}\end{array}
\]

Then $\phi(u_{m-1} z) = 0$ because there is a defect at the $n$'th position and $u_{m-1} \phi(z) = 0$ because $\phi(z)$ has defects at both position $m-1$ and $m$, so $u_{m-1}$ adds a link which means the result is zero in $V_{n-1,l-1}$. □
Proposition 2.22. When $q^{2(n-2l+1)} \neq 1$, the exact sequence in proposition 2.21 splits, which means

$$V_{n,l} \cong V_{n-1,l} \oplus V_{n-1,l-1}.$$  

Proof. This proposition can be proven by examining the eigenvalues of a central element $F_n$ in $D_n$. $F_n$ has two distinct eigenvalues when $q^{2(n-2l+1)} \neq 1$ whose eigenspaces each correspond to either $V_{n-1,l}$ or $V_{n-1,l-1}$. Because $F_n$ is central these eigenspaces are submodules of $V_{n,l}$ and then the sequence splits. For further details on $F_n$ we refer to [5].

If we assume that $q^{2(n-2l+1)} \neq 1$ we will have a splitting $\theta : V_{n,l} \rightarrow V_{n-1,l} \oplus V_{n-1,l-1}$. If we now order the basis of link states of $V_{n,l}$ such that those of $V_{n,l}$ come first, $\theta$ can be chosen such that it is represented by a matrix of the form

$$U_{n,l} = \begin{pmatrix} \text{id} & V_{n,l} \\ 0 & \text{id} \end{pmatrix}.$$  

Lemma 2.23. If a splitting $\theta$ exists we can define a bilinear form on $V_{n-1,l} \oplus V_{n-1,l-1}$ by

$$\langle x + x', y + y' \rangle = (\theta^{-1}(x + x'), \theta^{-1}(y + y'))_{n,l} \text{ for } x, y \in V_{n-1,l} \text{ and } x', y' \in V_{n-1,l-1}.$$  

Because this form is symmetric and invariant in the sense of lemma 2.12,

$$\langle x + x', y + y' \rangle = \langle y, z \rangle_{n-1,l} + \alpha_{n,l}(x', y')_{n-1,l-1},$$

for some $\alpha_{n,l} \in \mathbb{C}$.

Proof. We will prove this using the following argument. A bilinear form induces a map from its module $V$ to its dual by $v \mapsto \langle v, \cdot \rangle$. Because the bilinear form is invariant we have that this map is an intertwiner. When $V$ is irreducible its dual is as well, from Schur’s lemma it then follows that the induced maps are determined apart from a scalar, so they form a one-dimensional vector space, therefore the invariant bilinear forms do as well.

In this case we have that $V$ is the direct sum of two non-isomorphic irreducible modules, which means we have a two-dimensional space of bilinear forms. Because we can compare the form on $V_{n,l-1}$ with that on the direct sum we have that the factor before $\langle y, z \rangle_{n-1,l}$ is one.

When we view this in matrix form this becomes

$$G_{n-1,l} \oplus G_{n-1,l-1} = (U^{-1}_{n,l})^T G_{n,l} U^{-1}_{n,l} \Rightarrow G_{n,l} = U_{n,l}^T \begin{pmatrix} G_{n-1,l} \\ 0 \end{pmatrix} \alpha_{n,l} G_{n-1,l-1} U_{n,l}.$$  

This gives us the following recurrence relation to determine $\det G_{n,l}$ because $\det U_{n,l} = 1$,

$$\det G_{n,l} = \alpha_{n,l}^{d_{n-1,n-2l+1}} \det(G_{n-1,l}) \det(G_{n-1,l-1})$$

Proposition 2.24. When $q$ is not a root of unity $\alpha_{n,l} \neq 0$.

Proof. The determining of $\alpha_{n,l}$ is performed in [5], but because we are only interested in the fact that it is non-zero when $q$ is not a root of unity we will not do this and simply use this result.
Corollary 2.25. When $q$ is not a root of unity $\det G_{n,l} \neq 0$.

Proof. Note that

$$\det G_{n,0} = 1 \quad \text{and} \quad \det G_{2l-1,l} = 1.$$ 

The first follows because $V_{n,0}$ is spanned by just the link state with $n$ defects. The second follows from putting $G_{2l,l} = \beta G_{2l-1,l-1}$ into the recurrence relation. $G_{2l,l} = \beta G_{2l-1,l-1}$ follows from the fact that the one defect in $V_{2l-1,l-1}$ gives rise to precisely one less closed loop than in $V_{2l,l}$.

If we combine these two with the recurrence relation and proposition 2.24 the result follows. \hfill \Box

Corollary 2.26. When $q$ is not a root of unity the $V_{n,l}$ are irreducible $D_n$ modules.

Theorem 2.27. When $q$ is not a root of unity, $D_n$ is a semisimple algebra, the set \{ $V_{n,l}$ \mid 0 \leq l \leq \lfloor n/2 \rfloor \} is a complete set of non-isomorphic irreducible modules and as a left module $D_n$ decomposes as

$$D_n \simeq \bigoplus_{0 \leq l \leq \lfloor n/2 \rfloor} (\dim V_{n,l}) V_{n,l}.$$ 

Proof. Combining corollary 2.26 and proposition 2.17 we have that the $V_{n,l}$ are non-isomorphic irreducible modules. Then combining proposition 2.14 and Wedderburn’s theorem the result follows. \hfill \Box
3. Dilute Temperley-Lieb algebra

The introduction of the dilute Temperley-Lieb algebra will be similar to that of the regular Temperley-Lieb algebra. We will start by defining the dilute \(n\)-diagram algebra and determining its dimension, we will give a set of generators and relations between these generators to define the dilute Temperley-Lieb algebra and show the equivalence between the two algebras.

A dilute \(n\)-diagram is similar to a regular \(n\)-diagram except that we allow an even number of points to not be connected to any other points, we call these points vacancies. There must be an even number of vacancies because each line connects precisely two points and there are always an even number of points in a dilute \(n\)-diagram.

We can also construct the dilute \(n\)-diagram algebra, which we will often denote as \(dD_n\). This is the complex span over all the possible dilute \(n\)-diagrams up to isotopy. Where the multiplication is the same as the regular \(n\)-diagram algebra except that if a vacancy is connected to a line the result is zero. We will now show two examples for \(n = 3\)

\[
\begin{align*}
\DILUTE_3 & = 0 , \\
\DILUTE_3 & = 0
\end{align*}
\]

We would now like to give a set of generators for this algebra, but in order to do this we shall first introduce the following useful notation. When there is a dashed line in one of the diagrams this represents a sum two diagrams one where there are vacancies on the endpoints of the dashed line and one where there is a line between the two endpoints. An example for \(n = 3\)

\[
\begin{align*}
\DASHED_3 & = \DASHED_3 + \DASHED_3 + \DASHED_3 + \DASHED_3
\end{align*}
\]

The reason this notation is useful becomes apparent when we give the identity element of the dilute \(n\)-diagram algebra. It is simply the diagram with dashed lines at every point, this is the identity because for every dilute \(n\)-diagram there is precisely one element in the sum of the identity where the result of multiplication is not zero and for that element it acts as the identity on that dilute \(n\)-diagram.

\[
\id_n = \DASHED_n
\]

We will now determine the dimension of the dilute \(n\)-diagram algebra. Similar to the the \(n\)-diagram algebra we will give an equivalence between the dilute \(n\)-diagram algebra and \((n, p)\)-link diagrams. For this we will first define the dilute \(n\)-link diagrams. This is a horizontal
line with $n$ points connected the same as regular $n$-link diagrams except that we allow points to be vacancies. We orient these such that the defects are always facing up. An example for $n = 6$

![Diagram]

**Definition 3.1.** $A_n$ is the vector space spanned by all the dilute $n$-link diagrams.

**Definition 3.2.** $H_{n,k}$, $0 \leq k \leq n$ is the subspace of $A_n$ spanned by all the dilute $n$-link diagrams with at most $k$ defects.

We will call $A_n$ and $H_{n,k}$ the *link modules*. The reason for this name will become apparent when we determine the modules of the dilute $n$-diagram algebra. Any linear combination of dilute link diagrams will be called a *dilute link state*.

We can now define a bijective map $\psi : dD_n \to H_{2n,0}$ as follows. For any dilute $n$-link diagram we connect the two horizontal lines on the left and reorient the rest of the diagram so that this line forms the bottom of the diagram. Then we extend this linearly and obtain the map. For example

![Diagram]

This map is bijective because this process can easily be reversed by separating the horizontal line in the middle for a dilute $2n$-link diagram with no defects and reorienting the leftmost horizontal line to obtain a dilute $n$-diagram.

We can then transform such a dilute $2n$-dilute link diagram into a regular link diagram with no defects by simply removing all the vacancies. For example

![Diagram]

Now we can easily determine the dimension of the dilute $n$-diagram algebra using the dimension of the regular $n$-diagram algebra.

**Theorem 3.3.** The dimension of the dilute $n$-diagram algebra is

$$
\dim dD_n = \sum_{i=0}^{n} \binom{2n}{2i} \dim TL_i = \sum_{i=0}^{n} \frac{1}{i+1} \binom{2i}{i} \binom{2n}{2i}.
$$

**Proof.** By taking $2m \leq 2n$ and choosing $2m$ points we can take the subset of the dilute $n$-diagrams where the diagrams have vacancies at precisely those points. Combining the previous two procedures we obtain the link basis in $V_{2(n-m),0}$. This does not depend on the choice of points. Combining this with the factor $\binom{2m}{2i}$ obtained from the possible choices we have that the dilute $n$-diagrams with $2m$ vacancies have dimension $\binom{2n}{2m} \dim V_{2(n-m),0}$. But combining lemma 2.3 and 2.4 in chapter 2 we have that $\dim V_{2(n-m),0} = \dim TL_{n-m}$ which completes the proof. \qed
3.1. Dilute Temperley-Lieb algebra

The dilute Temperley-Lieb algebra has been the subject of much study, but in spite of this there does not exist a formal definition. A set of generators and relations has been proposed [8] during a study of more general braid algebras but it was given without proof. In this chapter we will give a generating set of the dilute $n$-diagram algebra, give a set of relations based on the ones given in [8] to define the dilute Temperley-Lieb algebra and prove the equivalence between the dilute $n$-diagram algebra and the dilute Temperley-Lieb algebra.

We will now give a set of generators of the dilute $n$-diagram algebra, but unlike the regular $n$-diagram algebra where the algebra was generated by the identity and one type of generators, we will need multiple different types of generators. There are several generating sets but we will be using $\{a_i, a_i^l, b_i, b_i^l, e_j, x_j, i \in [1, n-1], j \in [1, n]\}$ where we make the following identifications.

$e_i = \text{Diagram 1}$, $x_i = \text{Diagram 2}$, $a_i = \text{Diagram 3}$, $a_i^l = \text{Diagram 4}$, $b_i = \text{Diagram 5}$, $b_i^l = \text{Diagram 6}$

Note that it is possible to replace all the $x_i$ with $\text{id}_n$ to create a smaller set of generators because $\forall i, x_i = \text{id}_n - e_i$. But we will find that it is convenient to use the $x_i$’s.

**Theorem 3.4.** The set $\{a_i, a_i^l, b_i, b_i^l, e_j, x_j, i \in [1, n-1], j \in [1, n]\}$ is a generating set of the dilute $n$-diagram algebra.

**Proof.** We will prove this by first giving for every two points a diagram that is a product of generators that contains a line between those two points. Given two points in a dilute $n$-diagram we will label them by their horizontal position $p_i$ and $p_k$ where we assume without loss of generality that $i \geq k$. Now there are five options for which of the horizontal lines each point is on.

1. Both points are on the bottom horizontal line. Then $a_{i-1}^l \cdots a_{k+1}^l b_k^l$ has a line that connects the two points.
2. The two points are on different horizontal lines, \( i \neq k \). and \( p_i \) is on the bottom horizontal line. Then \( a_{i-1}^i \cdots a_k^i \) has a line that connects the two points.

\[
a_{i-1}^i \cdots a_k^i = \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
1 & k & \cdots & n
\end{array}
\]

3. The two points are on different horizontal lines and \( i = k \). Then simply \( e_i \) will suffice.

4. The two points are on different horizontal lines, \( i \neq k \) and \( p_i \) is on the top horizontal line. Then \( a_k \cdots a_{i-1} \) has a line that connects the two points.

\[
a_k \cdots a_{i-1} = \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
1 & k & \cdots & n
\end{array}
\]

5. Both points are on the top horizontal line. Then \( b_k a_{k+1} \cdots a_{i-1} \) has a line that connects the two points.

\[
b_k a_{k+1} \cdots a_{i-1} = \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
1 & k & \cdots & n
\end{array}
\]

Note that the diagrams constructed this way, which we will now refer to as *standard strings*, have dashed lines on all horizontal positions outside of the interval \([k, i]\). Where we note that these dashed lines act as the identity on their respective horizontal positions which ensures that these standard strings only act non-trivially on points in the interval \([k, i]\). Now given a dilute \( n \)-dilute diagram we can replace each of the lines with one of the standard strings, but we cannot simply concatenate these because the dilute \( n \)-diagram algebra is not commutative. An example for \( n = 3 \).

Now we will give an order of compositions which will ensure we end up with the desired diagram. We will do this as follows, we first give the five types of lines the order \( 1 < 2 < 3 < 4 < 5 \). Let \( x \) and \( y \) be two lines of the same type. If \( x \) and \( y \) are type 1 or 4 we say \( x < y \) if \( y \) has the point furthest to the left. If \( x \) and \( y \) are type 2, 3 or 5 we say \( x > y \) if \( y \) has the point furthest to the left. This totally orders all the lines for any dilute \( n \)-diagrams. We will write the standard strings corresponding to the lines such that the standard strings, for
which the lines are higher in the order, are to the right. We will say that a standard string is above or below another standard string if it is higher or lower respectively in the order, this terminology is used because when viewing the diagrams standard strings higher in the order are above ones lower. An example of a decomposition for $n = 6$,

We find that $d < c < b < a$. Where for each of the lines we have:

$$a = b_3a_4, \quad b = a_2a_3a_4a_5, \quad c = a_5^t a_4^t b_4^t \quad \text{and} \quad d = b_4^t.$$

When we combine these things we obtain $b_4^t a_5^t a_4^t b_5^t a_2a_3a_4a_5b_4a_4$ as a way of writing the diagram as a monomial in the generators. If we do so we obtain the following diagram as a result:

Note that we still have a dashed line where there should be vacancies. This brings us to the final part of the construction. For every vacancy on the top row we add the corresponding $x_i$ on the right and for the vacancies on the bottom row we place the corresponding $x_i$ to the left, note that we only need to add the $x_i$ when we have a dashed line in the construction, but for which vacancies this occurs is not always obvious so we add $x_i$ to all the vacancies to ensure that we always get the desired result.

All that remains is to prove that this construction works as intended. Recall that every standard string acts like the identity on all point on horizontal positions not on or between its endpoints. We will say that two lines overlap if a horizontal position exist where both of the corresponding standard strings do not act like the identity. Since we only construct lines when they exist in the diagram it suffices to prove that for every standard string in the construction, the line in it is only ever connected to dashed lines when we construct the diagram. Which means that for endpoints on the top horizontal row we need to check that the standard strings corresponding to lines above it are the identity there and for endpoints on the bottom row we need to check that all standard strings corresponding to lines below it are the identity on that horizontal position.

Standard strings corresponding to lines of type 3 only act on one horizontal position and no overlap can exist with other lines because otherwise we would have two crossing lines. Therefore the line in the standard string can only be connected to dashed lines.

For lines of type 1 we note that its endpoints are only on the bottom horizontal row. Therefore we only need to check that it works for lines of type 1 below it. But the standard string corresponding to such a line can only act as the identity on the endpoints of the first line or else they would overlap in the original diagram.

The same argument holds for lines of type 5 except that all the direction are reversed.

For lines of type 2 we note that for the endpoint on the top row only standard strings corresponding to lines of type 1 or 2 can possibly not be the identity there, but such lines are below it. For the endpoint on the bottom row the only types of lines for which the
corresponding standard strings can be different from the identity on that point are of type 2 or 5. But once again such lines must be above it.

A similar argument holds for lines of type 4.

The final step in the construction where we add the vacancies simply ensures that we are not left with any dashed lines in the final construction. The position in the construction clearly ensures that they do not connect to any line.

We now have a construction that for any possible dilute \( n \)-diagram gives a decomposition in the given generators.

We now want to use these generators to describe the dilute \( n \)-diagram algebra in terms of its characterizing relations between these algebraic generators.

**Definition 3.5.** For a \( \beta \in \mathbb{C} \), the dilute Temperley-Lieb algebra denoted \( dTL_n(\beta) \) or \( dTL_n \) is the algebra generated by \( \{ a_i, a_i^t, b_i, b_i^t, e_j, x_j \mid i \in [1, n-1], j \in [1, n] \} \), satisfying,

(i) Commutation relations

\[
O_iO_j = O_jO_i \quad \text{for} \quad |i - j| > 1,
\]

\[
O_ie_j = e_jO_i, \quad O_ix_j = x_jO_i \quad \text{for} \quad j \neq i, i + 1,
\]

\[
x_ix_j = x_jx_i, \quad e_ix_j = x_je_i, \quad e_ie_j = e_je_i.
\]

Where \( O_i \in \{ a_i, a_i^t, b_i, b_i^t \} \).

(ii) Projection relations

\[
e_i + x_i = \text{id}_n, \quad e_i^2 = e_i, \quad x_i^2 = x_i, \quad x_i e_i = 0.
\]

(iii) Occupation relations

\[
e_ia_i = a_i, \quad e_{i+1}a_i = 0, \quad a_ie_i = 0, \quad a_i e_{i+1} = a_i,
\]

\[
e_ia_i^t = 0, \quad e_{i+1}a_i^t = a_i^t, \quad a_i^te_i = a_i^t, \quad a_i^te_{i+1} = 0,
\]

\[
e_ib_i = 0, \quad e_{i+1}b_i = 0, \quad b_ie_i = b_i, \quad b_i e_{i+1} = b_i,
\]

\[
e_i b_i^t = b_i^t, \quad e_{i+1}b_i^t = b_i^t, \quad b_i^te_i = 0, \quad b_i^te_{i+1} = 0.
\]

The vacancy relations involving \( x_i \) follow from the occupation relations by \( x_i + e_i = \text{id}_n \).

(iv) Remaining relations

\[
a_i a_i^t = e_ix_{i+1}, \quad a_i^ta_i = x_ie_{i+1}, \quad b_ib_i^t = \beta x_ix_{i+1}, \quad b_{i+1}a_i = b_{i+1}x_i,
\]

\[
a_i^ta_i^t b_i^t = x_i b_i^{t+1}, \quad a_i a_{i+1} = b_{i+1} b_i^t, \quad a_i^t a_{i+1} = b_i^t b_i^{t+1}.
\]

**Proposition 3.6.** There exists a unique unit-preserving surjective algebra morphism \( \phi : dTL_n \to dD_n \) such that \( \phi \) maps the generators of \( dTL_n \) to the corresponding dilute \( n \)-diagrams of theorem 3.4.

**Proof.** To prove this we need to check that all the relations in \( dTL_n \) also hold in the dilute \( n \)-diagram algebra with the given identifications. This is simply a matter of writing the
diagrams on both sides of each relation. For example

\[ b_i a_{i+1} a_i = \]

\[ = b_{i+1} x_i. \]

Then \( \phi \) is surjective because of theorem 3.4.

From now on we will call the generators in \( \mathrm{dTL}_n \) letters and the products of these letters words. We are now going to prove that the dilute Temperley-Lieb algebra and the dilute \( n \)-diagram algebra are isomorphic. We will do this by showing that every word in the dilute Temperley-Lieb algebra is equal to a linear combination of elements in the dilute \( n \)-diagram algebra in the form we have previously seen in the proof of theorem 3.4. Before we prove this we shall first note a few things about the generators of \( \mathrm{dTL}_n \).

**Lemma 3.7.** The following relations hold in \( \mathrm{dTL}_n \).

\[ a_i b_{i+1}^i = a_{i+1}^i b_i^i \text{ and } b_i a_{i+1} = b_{i+1} a_i^i. \]

**Proof.**

\[ a_i b_{i+1}^i = (a_i x_i) b_{i+1} = a_i (a_{i+1}^i b_{i+1}^i) = e_i x_{i+1} a_i^i b_i^i = a_{i+1} e_i b_i^i = a_{i+1}^i b_i^i, \]

\[ b_i a_{i+1} = (b_i e_i) (a_{i+1} x_{i+1}) = b_i (a_{i+1} e_i x_{i+1}) = b_i a_{i+1} a_i^i = (b_i x_i) a_i^i = b_{i+1} a_i^i. \]

**Definition 3.8.** We define a word \( u \) in \( \mathrm{dTL}_n \) to be **irreducible** if it is not equal to a linear combination of words with fewer letters.

We note that the only relation that allows a word to be equal to a linear combination of at least two distinct words is \( e_i + x_i = \text{id}_n \). From this it follows that every word in \( \mathrm{dTL}_n \) is equal to some irreducible word and using relations that do not change the number of letters will preserve the irreducibility.

We now want to rewrite the irreducible words of \( \mathrm{dTL}_n \) to a form more suited for proving the equivalence between \( \mathrm{dTL}_n \) and the dilute \( n \)-diagram algebra. For this we are interested in which generators do not commute with one another.
We first note that both \( x_i \) and \( e_i \) occurring as letters in an irreducible word must commute with all other letters in the irreducible word. This is because if \( e_i \) or \( x_i \) does not commute with another letter we have a relation that reduces the word further.

**Proposition 3.9.** The following pairs of letters are all the possible pairs that do not commute and are non-zero,

\[
\begin{align*}
& a_{i+1} a_i \quad a_{i} a_{i-1} \quad a_{i-1} a_i \quad a_i a_{i+1} \quad b_i b_i \\
& b_{i+1} a_i \quad a_i b_{i-1} \quad b_{i} b_{i+1} \quad a_{i} b_{i+1} \quad b_i b_{i+1} \quad b_{i-1} b_i \quad b_i b_{i-1} \quad b_i b_{i+1} \quad b_{i-1} b_i \quad b_i b_{i+1}
\end{align*}
\]

where we note that some pairs occur multiple times in the diagram, this is so that it becomes somewhat easier to read.

**Proof.** Because of the commutation relation we only need to compare each letter at the \( i \)-th position with all the other ones on \( i \) and \( i \pm 1 \) on both the left and the right and checking whether they are equal to zero by using the occupation and vacancy relation. We will only show for one that it is equal to zero,

\[
a_i b_i = (a_i e_{i+1}) b_i = a_i (e_{i+1} b_i) = 0.
\]

\( \square \)

**Definition 3.10.** A **string** is a sequence of letters in a word in \( \mathfrak{dTL}_n \) consisting of \( a_i, a_i^t \) and at most one \( b_i \) or \( b_i^t \), that is not equal to zero and where every two consecutive letters do not commute. We will say that a letter commutes with a string if it commutes with every letter in the string.

Note that the standard strings defined in the proof of theorem 3.4 are strings in this definition.

**Theorem 3.11.** The dilute Temperley-Lieb algebra is isomorphic to the dilute \( n \)-diagram algebra.

**Proof.** We already have the surjective map \( \phi \) from proposition 3.6, now it suffices to prove that it is injective. We will prove this by showing that every word \( u \) in \( \mathfrak{dTL}_n \) can be written as a linear combination of words in the same form as the ones constructed in theorem 3.4. To show this we will inductively construct the standard strings in the proof of theorem 3.4, we start by changing a word \( u \) in \( \mathfrak{dTL}_n \) so that all the \( b_i \) are part of these standard strings. We then repeat this for the \( b_i^t \) and finally reorder the remaining \( a_i \) and \( a_i^t \) not in standard strings so that we obtain the construction in the proof of theorem 3.4, during this process we will refer to strings as completed when we no longer want to alter them, in other words when they precisely correspond to the desired standard string in the construction in the proof of theorem 3.4. Note that during the process all the strings we construct will be standard strings, but we will simply refer to them as strings. Also note that we will only discuss the \( x_i \) and \( e_i \) at the end of the proof because these commute with all other letters in an irreducible word.

Given an irreducible word \( u \) in \( \mathfrak{dTL}_n \), we will inductively choose \( b_i \) and move them to the right creating strings in the process. When choosing a \( b_i \) we view the set of all \( b_i \) with the
highest index $i$ in $u$ not in a completed string, we then choose the $b_i$ in this set that is the furthest to the left in $u$. When we now say choose a $b_i$ we mean selecting a $b_i$ in this manner. We will inductively create a string using the chosen $b_i$, we assume that the string has the form $b_ia_{i+1}\cdots a_{k-1}a_k$, we will then move this string to the right by commuting the string with each letter in $u$ with which it commutes. We continue this until we encounter a letter with which the string does not commute, there are nine options for the letter that do not result in zero.

(i) The string consists of just $b_i$ and it encounters $b'_i$, in this case we replace the string with $\beta x_i x_{i+1}$ and choose a new $b_i$.

(ii) $a_{k+1}$, in this case we simply add this letter to the string and continue with the process.

(iii) $a^t_{i-1}$, we first note that this commutes with every $a_i$ in the string so we can write
\[ b_ia_{i+1}\cdots a_{k-1}a_k a^t_{i-1} = (b_ia^t_{i-1})a_{i+1}\cdots a_{k-1}a_k = b_{i-1}a_{i+1}\cdots a_{k-1}a_k \]
and we choose a new $b_i$.

(iv) $b^t_{i-1}$, we first note that this commutes with every $a_i$ in the string so we can write
\[ b_ia_{i+1}\cdots a_{k-1}a_k b^t_{i-1} = b_ib_{i-1}a_{i+1}\cdots a_{k-1}a_k = a_{i-1}a_{i+1}\cdots a_{k-1}a_k \]
and we choose a new $b_i$ if it exists otherwise we end the process.

(v) $b^t_{k+1}$, we can rewrite the string as follows
\[ b_ia_{i+1}\cdots a_{k-1}a_k b^t_{k+1} = b_ia_{i+1}\cdots a_{k-1}a^t_{k+1} b^t_k = b_ia_{i+1}\cdots a^t_{k+1}a_{k-1}b^t_k \]
\[ = b_ia_{i+1}\cdots a^t_{k+1}a^t_{k-1}b^t_{k-1} = b_ia_{i+1}a_{k+1}a^t_k \cdots a_{i+2}b^t_{i+1} = a^t_{k+1}a^t_k \cdots a^t_{i+2}b^t_{i+1} \]
\[ = a^t_{k+1}a^t_k \cdots a^t_{i+2}a^t_{i+1}a_i \]
and we choose a new $b_i$ if it exists otherwise we end the process.

(vi) $a^t_k$ where $i \neq k$, we can rewrite the string as follows
\[ b_ia_{i+1}\cdots a_{k-1}a_k a^t_k = b_ia_{i+1}\cdots a_{k-1}a_k x_{k+1} = b_ia_{i+1}\cdots a_{k-1}x_{k+1} \]
and we continue the process with the new string $b_ia_{i+1}\cdots a_{k-1}$.

(vii) $a_i$ where $i \neq k$, we can rewrite the string as follows
\[ b_ia_{i+1}\cdots a_{k-1}a_k a_i = b_ia_{i+1}a_{i+2}\cdots a_{k-1}a_k = b_{i+1}x_ia_{i+2}\cdots a_{k-1}a_k = x_i b_{i+1}a_{i+2}\cdots a_{k-1}a_k \]
and we continue the process with the string $b_{i+1}a_{i+2}\cdots a_{k-1}a_k$.

(viii) $b_l$ with $l \in [i+1, k-2]$, we now say that the string $b_ia_{i+1}\cdots a_{k-1}a_k$ is completed and we choose a new $b_i$ if it exists otherwise we end the process.

(ix) There are no further elements to the right of the string, we then say that this string is completed and choose a new $b_i$ if it exists otherwise we end the process.
After completing this process the result is that $u = wv$ where $w$ is a word that does not contain any $b_i$’s and $v$ is of the following form

$$v = (b_1 a_{i_1 + 1} \cdots a_{k_1})(b_{i_2} a_{i_2 + 1} \cdots a_{k_2}) \cdots (b_{i_m} a_{i_m + 1} \cdots a_{k_m}).$$

where $i_1 < i_2 < \cdots < i_m$.

A similar argument can be made for $b_i^t$ except that the strings are of the form $a_{i_1}^t a_{i_1 - 1}^t \cdots a_{i_1 + 1}^t b_i^t$, we move the strings to the left and when choosing a $b_i^t$ from the set of all $b_i^t$ with the highest index $i$ not in a completed string, we choose the one that is the furthest to the right in $u$. This works because there is a symmetry in all the relations involving $b_i$ or $b_i^t$ which results in

$$u = (a_{j_1}^t a_{j_1 - 1}^t \cdots a_{j_1 + 1}^t b_{o_1}^t) \cdots (a_{j_p}^t a_{j_p - 1}^t \cdots a_{j_p + 1}^t b_{o_p}^t) w(b_{i_1} a_{i_1 + 1} \cdots a_{k_1}) \cdots (b_{i_m} a_{i_m + 1} \cdots a_{k_m}).$$

Where $i_1 < i_2 < \cdots < i_m$, $o_1 > o_2 > \cdots > o_p$ and $w$ only contains $a_i, a_i^t, x_i$ and $e_i$.

Next we will reorder $w$. For this we first note that for all $i,j$ $a_i a_j = 0$ or $a_i a_j = a_j a_i$. This means that in $w$ for all $i$, $a_i$ commutes with everything but $a_{i-1}$ and $a_{i+1}$. When choosing $a_i$ we will view the set of $a_i$ with the lowest index $i$ not in a completed string in $u$, we then choose the $b_i$ in this set that is the furthest to the left. We will now inductively create a string $a_i a_{i+1} \cdots a_k$ by moving the string to the right until it does not commute with an element in $w$. For such an element there are three options

(i) $a_{i-1}$, this is not possible by assumption.

(ii) $a_j$ with $j \in [i, k-1]$, this is not possible because we would have

$$a_i a_{i+1} \cdots a_k a_j = a_i a_{i+1} \cdots a_k (e_j a_j) = a_i a_{i+1} \cdots a_j e_j a_{i+1} \cdots a_k a_j = 0$$

and we know that $w$ is also irreducible.

(iii) $a_{k+1}$, we continue with the string $a_i a_{i+1} \cdots a_k a_{k+1}$.

This continues until we reach the end of $w$ or a completed string, we then say $a_i a_{i+1} \cdots a_k$ is a completed string and choose a new $a_i$. This process ends when all the $a_i$ are part of completed strings. After this process ends we have

$$w = z(a_{i_1} a_{i_1 + 1} \cdots a_{k_1}) \cdots (a_{i_m} a_{i_m + 1} \cdots a_{k_m}).$$

Where $z$ consists solely of $x_i, e_i$ and $a_i^t$ and $i_1 > i_2 > \cdots > i_m$. We can again use a similar argument for the $a_i^t$ except that we move the strings to the left. We then obtain

$$w = (a_{i_1}^t a_{i_1 - 1}^t \cdots a_{k_1}^t) \cdots (a_{i_m}^t a_{i_m - 1}^t \cdots a_{k_m}^t) z(a_{i_{m+1}} a_{i_{m+1} + 1} \cdots a_{k_{m+1}}) \cdots (a_{i_p} a_{i_p + 1} \cdots a_{k_p}).$$

where $z$ only consists of $e_i$ and $x_i$, $k_1 < k_2 < \cdots < k_m$ and $i_{m+1} > i_{m+2} > \cdots > i_p$. We now have a reordering of the irreducible word $u$ in strings. We note that apart from the standard strings of type 3, namely the $e_i$, the order these string are in precisely corresponds to the order of the corresponding standard strings in the proof of theorem 3.4. All that remains is to consider the remaining $x_i$ and $e_i$. We could simply reorder the $e_i$ to be in the desired order, but one problems remains namely that the image of any given word $u \in dTL_n$ might contain dashed lines. For this we will now introduce a linear spanning set of $dTL_n$, which we will denote $X_n$. We say that a word $u \in X_n$ if for all $i$ one of four cases applies,

$$e_i u e_i = u \text{ or } e_i u x_i = u \text{ or } x_i u e_i = u \text{ or } x_i u x_i = u.$$
To prove that $X_n$ is a linear spanning set first note that $\text{id}_n = x_i + e_i$ for all $i$. But then for any word $u$ in $\text{dTL}_n$

$$u = (x_1 + e_1) \cdots (x_n + e_n)u(x_1 + e_1) \cdots (x_n + e_n).$$

We can then write this product into a sum of elements of $X_n$ thus proving that it forms a linear spanning set. The reason we are interested in $X_n$ is because the elements precisely correspond to the dilute $n$-diagrams with no dashed lines under the map $\phi$.

We know that every word $u$ in $\text{dTL}_n$ is equal to an irreducible word in strings such that these strings have the same order as the standard strings in the proof of theorem 3.4. We know that all the $e_i$ and $x_i$ in $u$ commute with every other letter. This means we can also reorder the $e_i$ so that they have the same order as the corresponding standard strings and we can place the $x_i$ on the outside of the word $u$ and add all the $x_i$ to the ends of $u$ that act as the identity on it and remove all the $x_i$’s that can be removed solely with the commutation relations and $x_i^2 = x_i$, this last step with the $x_i$ precisely corresponds to the last step of the proof of theorem 3.4 where we add the vacancies to the diagram.

We now have a linear spanning set for $\text{dTL}_n$ for which each element precisely corresponds to the construction of a unique dilute $n$-diagram. This proves the equivalence between $\text{dTL}_n$ and the dilute $n$-diagram algebra.  

\[ \square \]

3.2. Standard modules of the dilute $n$-diagram algebra

Earlier in this chapter we have introduced the dilute $n$-link diagrams and there exists a natural left action from the dilute $n$-diagrams onto these dilute link diagrams by concatenating the two and removing the original points of the link diagram. Then replacing each line not connected to any of the remaining points with a factor $\beta$ if it is closed, a factor 0 if it is connected to a vacancy and simply removing it if neither is the case. The remaining lines will all be connected to at least one of the points. If such a line is connected to a vacancy it is replaced with a zero, if it is connected to a defect it becomes a defect otherwise it remains a link. Finally any vacancies not on any of the remaining points are removed. A few examples for $n = 4$
If we extend the action linearly we find that $A_n$ is a left module of the dilute $n$-diagram algebra. Note that this action can only ever decrease the number of defects. This means that $H_{n,k}$ is a submodule of $A_n$. We have the natural inclusions

$$H_{n,0} \subset H_{n,1} \subset \cdots \subset H_{n,n} = A_n.$$ 

Any linear combination of dilute link diagrams will be called a *dilute link state*.

**Definition 3.12.** The standard modules $S_{n,k}$ are the quotients of two consecutive link modules namely

$$S_{n,k} \simeq H_{n,k}/H_{n,k-1} \quad \text{for} \quad 1 \leq k \leq n \quad \text{and} \quad S_{n,0} = H_{n,0}.$$ 

Note that the elements of each $S_{n,k}$ are generated by equivalence classes with one unique dilute $n$-link diagram with $k$ defects. For that reason we will simply denote the equivalence classes by that unique dilute link diagram.

We will now let $Y_{n,k}$ be the set of all dilute $n$-link diagrams with $k$ defects. This set can be viewed as a basis for $S_{n,k}$ if we view the each diagram as representing the corresponding equivalence class.

**Definition 3.13.** We define the injective bilinear map $|\cdot, \cdot|_{n,k} : S_{n,k} \times S_{n,k} \to \mathbb{D}_n$ as follows. For $u, v \in Y_{n,k}$ we let $|u\bar{v}|$ be the $n$-diagram formed by viewing the dilute link diagram $u$ as the bottom row in a dilute $n$-diagram, the reflected state of the dilute link diagram $v$ as the top row and connecting each defect with one on the opposite row. We then extend this linearly to $S_{n,k} \times S_{n,k}$ to obtain the map.
Note that connecting the defects is unique because there is only one way so that none of the lines cross. Now an example for $n = 4$ and $k = 2$,

It will now be useful to view the subset $X_{n,k}$ of $Y_{n,k}$ consisting of all the dilute link diagrams with $k$ defects and $n - k$ vacancies.

**Lemma 3.14.** Let $z \in X_{n,k}$ and $u, v$ be any dilute link diagrams in $S_{n,k}$ then the following properties hold:

(i) $|u\bar{v}|z = 0$ when $z \neq v$.

(ii) $|u\bar{v}|z = u$ when $z = v$.

**Proof.** Suppose $v \neq z$, in the concatenation of $|u\bar{v}|$ with $z$ we identify the points of $\bar{v}$ with those of $z$. Since $z$ has precisely $n - k$ vacancies the result of this concatenation is zero unless every vacancy of $z$ is identified with one of $v$, but we know that $v$ has $k$ defects which in this case must all be connected to a defect of $z$, but then $z = v$. Therefore we can conclude that $|u\bar{v}|z = 0$ when $z \neq v$.

Suppose $v = z$, we first note that each of the defects of $u$ is still a defect in $|u\bar{v}|z$ because it is connected to two defects. Furthermore it is the case that every vacancy on $\bar{v}$ and $z$ is identified with another vacancy. Which means that $|u\bar{v}|z = u$. 

\[\square\]

**Proposition 3.15.** For the standard modules $S_{n,k}$ the following properties hold,

(i) $S_{n,k}$ is cyclic and every non-zero element of $\text{span}X_{n,k}$ is a generator.

(ii) $S_{n,k}$ is indecomposable.

(iii) $S_{n,k} \simeq S_{n,j} \iff i = k$.

**Proof.** From lemma 3.14(ii) it follows that any element in $X_{n,k}$ is a generator for $S_{n,k}$. Then for any non-zero element $v$ in $\text{span}X_{n,k}$ we know that $v$ is the linear combination of elements in $X_{n,k}$. Let $w \in X_{n,k}$ be a dilute link diagram such that $w$ is a non-zero component of $v$. Then for all $u \in S_{n,k}$, $|u\bar{v}|v = cu$ for some $c \in \mathbb{C}^\times$.

(ii) Suppose that $S_{n,k} \simeq A \oplus B$ for some non-trivial submodules $A$ and $B$. Let $z \in X_{n,k}$, we know that $z$ generates $S_{n,k}$ which means it cannot be an element of either $A$ or $B$, therefore $z = a + b$ for some non-zero link states $a \in A$ and $b \in B$. We know that $z = |z\bar{z}|z = |z\bar{z}|(a + b)$, then $|z\bar{z}|a \in A$ and $|z\bar{z}|b \in B$. Suppose $|z\bar{z}|a$ is zero, then $|z\bar{z}|b = z \in B$ and $B = S_{n,k}$. Suppose $|z\bar{z}|a$ is non-zero then it must have a non-zero component along $z$ in the basis of dilute link diagrams, then from lemma 3.14 it follows that $|z\bar{z}|a = ca$ for some $c \in \mathbb{C}^\times$. Which implies that $A = S_{n,k}$. From this it follows that $S_{n,k}$ is indecomposable.

(iii) The statement "$\iff"$ is clearly trivial. Suppose $S_{n,k} \simeq S_{n,j}$ where we let $k \leq j$. We know there exists a module isomorphism $\theta : S_{n,k} \to S_{n,j}$. Let $z \in X_{n,k}$ and $u \in S_{n,k}$ such that $|u\bar{z}|$ is non-zero, so $0 \neq \theta(|u\bar{z}|z) = |u\bar{z}|\theta(z)$. This means that $\theta(z)$ has $n - k$ vacancies, but $k \leq j$ so the remaining points must be occupied by defects, therefore $j = k$. 

\[\square\]
Proposition 3.16. The dimension of $S_{n,k}$ is given by,

$$\dim S_{n,k} = \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n}{k+2l} \dim V_{k+2l,l},$$

where we recall that $\dim V_{n,l} = \binom{n}{1} - \binom{n}{l-1}$. 

Proof. We know that the set of dilute $n$-link diagrams with $k$ defects is a basis of $S_{n,k}$ when we view them as equivalence classes. Recall that there exists a correspondence between these dilute link diagrams and the regular link diagrams by removing the vacancies. Also note that the number of vacancies in such dilute link diagrams can only vary by even numbers because only by adding or removing links can this number be altered.

Next, order these dilute link diagrams by the number of links, then for a set of $n$-link diagrams with $k$ defects and $l$ links the size is the same as the dimension of $V_{n,l}$ apart from a factor of $\binom{n}{k+2l}$ that comes from choosing the places of the vacancies. If we then sum over all the possible number of links we obtain the dimension of $S_{n,k}$. \hfill \qed

Proposition 3.17. The dimension of the dilute $n$-diagram algebra is also given by

$$\dim dD_n = \sum_{k=0}^{n} (\dim S_{n,k})^2.$$ 

Proof. Recall that for every $k \in \{0, 1, \cdots, n\}$ we have an injective bilinear map $|\cdot, \cdot|_{n,k} : S_{n,k} \times S_{n,k} \rightarrow \text{dilute } n\text{-diagram algebra}$. First note for every dilute $n$-diagram $u$ there are two dilute $n$-link diagrams $v$ and $w$ such that $u = [v \bar{w}]$. These dilute $n$-link diagrams can be obtained by identifying $v$ with the bottom row of $u$ and $\bar{w}$ with the top row and letting all the lines that connect the two rows be defects.

Also note that every dilute $n$-diagram in the image of $|\cdot, \cdot|_{n,k}$ has precisely $k$ lines that connect the two horizontal rows, therefore each dilute $n$-diagram corresponds to two dilute $n$-link diagrams and the result follows. \hfill \qed

3.3. Irreducibility of the standard modules

We now have a set of modules for the dilute $n$-diagram algebra and we want to explore whether these modules are irreducible. If this were the case proposition 3.17 and Wedderburn’s theorem would imply that the standard modules are all the possible modules. For this we will introduce a bilinear form for the dilute $n$-link diagrams. We will then view the matrix representing this bilinear form and using the results for the regular $n$-diagram algebra we will show that in most cases the link modules are irreducible.

Definition 3.18. The bilinear form $\langle \cdot, \cdot \rangle_{n,k}$ is defined on $S_n$ as follows. If $x$ and $y$ are two dilute $n$-link diagrams with $k$ defects, $\langle x, y \rangle_{n,k}$ is computed by concatenating the reflected link state of $x$ with $y$. If not every defect is connected with a defect on the opposite side or a vacancy is connected to a line, $\langle x, y \rangle_{n,k} = 0$. Otherwise $\langle x, y \rangle_{n,k} = \beta^m$ where $m$ equals the number of closed loops.
A few examples for \( n = 4 \) and \( k = 1 \)

\[
\langle \begin{array}{c}
\bullet \\
\end{array} , \begin{array}{c}
\bullet \\
\end{array} \rangle_{n,1} \rightarrow \begin{array}{c}
\emptyset \\
\end{array} \rightarrow \beta
\]

\[
\langle \begin{array}{c}
\bullet \\
\end{array} , \begin{array}{c}
\circ \\
\end{array} \rangle_{n,1} \rightarrow \begin{array}{c}
\emptyset \\
\end{array} \rightarrow 1
\]

\[
\langle \begin{array}{c}
\bullet \\
\end{array} , \begin{array}{c}
\circ \\
\end{array} \rangle_{n,1} \rightarrow \begin{array}{c}
\emptyset \\
\end{array} \rightarrow 0
\]

This bilinear form is an extension of the one we defined for regular \( n \)-diagram algebra. This bilinear form is also symmetric because switching the two link states results in a horizontal reflection when written in diagrams.

**Definition 3.19.** We will call \( x, y \in S_{n,k} \) *orthogonal* if \( \langle x, y \rangle_{n,k} = 0 \).

**Lemma 3.20.** If \( x, y \in S_{n,k} \) and \( U \in dD_n \) then

\[
\langle x, Uy \rangle_{n,k} = \langle U^\dagger x, y \rangle_{n,k}
\]

where \( U^\dagger \) is the diagram obtained by reflecting \( U \) across its horizontal axis, if \( U \) is a linear combination of diagrams each of those diagrams is reflected across its horizontal axis.

**Proof.** This immediately follows from writing both sides in diagrams.

**Lemma 3.21.** If \( x, y, z \in S_{n,k} \), then

\[
|x\bar{y}|z = (y, z)_{n,k}x.
\]

**Proof.** It suffices to check this for when they are all dilute link diagrams because of linearity. \( |x\bar{y}|z \neq 0 \) only if all defects and vacancies of \( z \) are linked to defects and vacancies of \( y \). But this is precisely the case when \( (y, z)_{n,k} \neq 0 \). Furthermore all defects, arcs and vacancies of \( x \) will remain the same. This means that \( |x\bar{y}|z \) is proportional to \( x \) and this proportionality constant is precisely \( (y, z)_{n,k} \).

In proposition 3.15(ii) we have seen that every non-zero element in \( \text{span}X_{n,k} \) is a generator of \( S_{n,k} \) but we can now use the previous lemma to find a much larger group of generators.

**Lemma 3.22.** \( x \in S_{n,k} \) is a generator of \( S_{n,k} \) if there exists \( y \in S_{n,k} \) such that \( (y, z)_{n,k} \neq 0 \).
**Proof.** Let \( y \in S_{n,k} \) such that \( \langle y, x \rangle_{n,k} = \alpha \neq 0 \). Then for any \( z \in S_{n,k} \) it is the case that \( z \) and \( y \) have the same number of defects and therefore \( |z\bar{y}| \) is an element in the dilute \( n \)-diagram algebra. Then \( \frac{1}{\alpha}|z\bar{y}|x = \frac{1}{\alpha}\langle y, z \rangle_{n,k}z = z \) therefore \( x \) is a generator of \( S_{n,k} \).

We now know that any link state not orthogonal to all links states is a generator of \( S_{n,k} \).

**Definition 3.23.** The dilute radical \( R_{n,k} \) of \( S_{n,k} \) is defined to be,

\[
R_{n,k} = \{ x \in S_{n,k} \mid \langle y, z \rangle_{n,k} = 0 \text{ for all } y \in S_{n,k} \}.
\]

The usefulness of \( R_{n,k} \) becomes immediately apparent from the following lemma.

**Lemma 3.24.** \( R_{n,k} \) is the maximal submodule of \( S_{n,k} \).

**Proof.** The fact that \( R_{n,k} \) is a submodule follows immediately from the linearity of the bilinear form. That \( R_{n,k} \) is also maximal is a direct consequence of lemma 3.22. 

Similar to the regular \( n \)-diagram algebra we want to determine the dilute radical by viewing the matrix representing the bilinear form.

**Definition 3.25.** The dilute Gram matrix \( dG_{n,k} \) is the matrix representing the bilinear form \( \langle \cdot, \cdot \rangle_{n,k} \) in the basis of dilute link diagrams.

**Lemma 3.26.** The dilute radical \( R_{n,k} = \ker dG_{n,k} \).

**Proof.** Follows immediately from the definition of \( dG_{n,k} \). 

**Proposition 3.27.** The dilute Gram matrix for \( S_{n,k} \) is

\[
dG_{n,k} = \bigoplus_{l=0}^{[\frac{n-k}{2}]} \binom{n}{k+2l} G_{k+2l,k},
\]

where we recall that \( G_{k+2l,k} \) is the Gram matrix for \( V_{n,l} \) and as such \( R_{n,k} \neq \{0\} \) if \( \det dG_{n,k} \neq 0 \).

**Proof.** We first note that for any two dilute link diagrams in \( S_{n,k} \) the bilinear form is only non-zero if they have the same number of vacancies and the vacancies coincide. If we order \( dG_{n,k} \) by the number of vacancies first and by the position of these vacancies second, \( dG_{n,k} \) becomes a block-diagonal matrix where each of the blocks consists of the products of dilute link diagrams with the same vacancies. But each of these blocks then coincides with the Gram matrix with the same number of links and \( k \) defects. Which gives us the desired result.

**Corollary 3.28.** The determinant of the dilute Gram matrix \( dG_{n,k} \) is given by

\[
\det G_{n,k} = \prod_{l=0}^{[\frac{n-k}{2}]} (\det G_{k+2l,k})^{(\frac{n}{k+2l})}.
\]

**Corollary 3.29.** When \( q \) is not a root of unity \( S_{n,k} \) is irreducible as a left \( dD_{n} \)-module.

**Proof.** Combining corollary 2.25 and 3.28 we know that \( \det dG_{n,k} \neq 0 \) if \( q \) is not a root of unity. This means that \( R_{n,k} = \{0\} \), but from lemma 3.24 we know that it is the maximal submodule. Therefore \( S_{n,k} \) is irreducible.
\textbf{Theorem 3.30.} If $q$ is not a root of unity, the dilute $n$-diagram algebra is semisimple, the set $\{S_{n,k}|0 \leq k \leq n\}$ is a complete set of non-isomorphic irreducible modules and as a left module the dilute $n$-diagram algebra decomposes as

$$dD_n = \bigoplus_{0 \leq k \leq n} (\dim S_{n,k})S_{n,k}.$$ 

\textit{Proof.} Combining corollary 3.29 and proposition 3.15 we know that the set $\{S_{n,k}|0 \leq k \leq n\}$ is a set of non-isomorphic irreducible modules. Then combining proposition 3.17 and Wedderburn’s theorem the rest follows. \qed
4. Physical application of the Temperley-Lieb algebra

In this chapter we will explore the relation between the Temperley-Lieb algebra and various physical models. For this we will first start with a short introduction of some of the basics of statistical physics such as the partition function.

For any state $x$ of some system, the Boltzmann weight is given by $e^{-\beta E(x)}$. Here $E(x)$ is the energy of the state $x$ and $\beta = \frac{1}{k_B T}$, where $k_B$ is Boltzmann’s constant and $T$ is the temperature. We note that this $\beta$ is not to be confused with the $\beta$ we have previously seen in the (dilute) Temperley-Lieb algebra, but it should always be clear which of the two is used. Then the partition function $Z$ is given by

$$Z = \sum_x e^{-\beta E(x)},$$

where we sum over all the possible states $x$ of our system. It can be shown that the probability of any state $x$ is given by

$$P(x) = \frac{1}{Z} e^{-\beta E(x)}.$$

4.1. Transfer matrix

The models we will now consider will have for states which consist of paths on a square lattice. These paths can be generated by a transfer matrix which we will define using the following two links,

We give the first link the weight $a$ and the second one the weight $b$. If we view these links as operators acting on two sites of a link diagram we can let a row of them act on a link diagram which will be how we define the transfer matrix.

If we have a line with $n$ vertices we can construct a transfer matrix in two steps. First we view the rows with links at every odd position. Then we have $W$ which is the sum over all these rows with their corresponding weights. An example of such a row for $n = 7$

However we note that this does nothing with the last vertex, so we let it simply act as the identity there which is a line connecting it with the same vertex on the next row. The next step will be viewing $V$ which is the sum over all the rows with links at the even positions where again each row has the corresponding weight. An example of such a row combining with the previous row in $W$
Now the transfer matrix $T$ is simply the product of $V$ and $W$. We note that the transfer matrix is a sum over rows in $V$ combined with rows in $W$ with the corresponding weights and each of the elements in this sum corresponds to a $n$-diagram because it connects $n$ vertices on two lines with non-intersecting lines. Of course not every $n$-diagram corresponds to such an element. The previous example would correspond to

Since each element in transfer matrix corresponds to an $n$-diagram we can concatenate multiple of these in the same way as in the Temperley-Lieb algebra. An example for this concatenation.

The element in the Temperley-Lieb algebra that corresponds to this term in the product of transfer matrices is

4.2. Potts model

We will now define the $p$-state Potts model and show its relation on square lattices with the (dilute) Temperley Lieb algebra. There exist multiple Potts models but we will be using the ‘scalar’ model as opposed to the ‘vector’ model. The Potts model has various applications due to its simplicity. It is the basis for models of phase transitions that break the symmetric group $S_p$.

A lattice $\mathcal{L}$ consists of $N$ sites and lines between sites to denote that they are neighbours. Then for any lattice $\mathcal{L}$ we associate with each site $i$ a spin $\sigma_i \in \{1, 2, \cdots, p\}$. We let neighboring spins have energy $-\epsilon$ if they are alike and zero if they are different. Then any state $x$
of this model is one of the possible configuration of spins on this lattice and the energy for a state \( x \) is

\[
E(x) = \epsilon \sum_{(i,j)} \delta(\sigma_i, \sigma_j),
\]

where we sum over all the neighbouring sites \((i, j)\). Then the partition function is

\[
Z = \sum_x \exp \left( \beta \epsilon \sum_{(i,j)} \delta(\sigma_i, \sigma_j) \right).
\]

Where we sum over \( p^N \) states. We will now set

\[
v = e^{\beta \epsilon} - 1,
\]

then we get,

\[
Z = \sum_x \prod_{(i,j)} (1 + v \delta(\sigma_i, \sigma_j)).
\]

From now on we will only consider square lattices which simply consist of a \( M \times N \) grid. With each square lattice \( L \) we can associate a surroundig lattice \( L' \) by taking the sites of \( L' \) to be the midpoints of edges in \( L \), we will call these the 'internal' sites. There are edges between two sites in \( L' \) only if the corresponding edges in \( L \) have a site in common. Finally we add sites at the boundary such that each site in \( L \) is contained within a square in \( L' \), we will call these the 'external' sites. We will find it useful to shade the squares which contain sites in \( L \). An example for a \( 3 \times 4 \) grid.

We will now define a so-called ice-type model on this lattice \( L' \) and claim that its partition function is \( p^{-N/2} Z \). This is called an ice-type model because a similar model can be used to describe ice. First let \( \theta \) and \( z \) be the parameters given by

\[
p^{1/2} = 2 \cosh \theta, \quad s = e^{\theta/2}.
\]

Then the ice-type model is defined as follows.

(i) We place arrows on each edge of \( L' \) such that at each site an equal number of arrows point in and out.

(ii) With each external site we associate a weight \( s^{1/2} \) if the arrows turn left through the site and a weight \( s^{-1/2} \) if they turn right.
(iii) For the internal sites there are six possible configurations.

\[ \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \]

These have the weights \( \omega_1 = 1, \omega_2 = 1, \omega_3 = x, \omega_4 = x, \omega_5 = s^{-1} + xs, \omega_6 = s + xs^{-1} \).

Note that some of the internal sites need to be rotated 90°

Then the partition function becomes

\[ Z' = \sum \prod (\text{weights}). \]

Where the sum is over all possible configurations of the arrows and the product is over all the sites.

In [3] it is shown that \( Z' = p^{-1/2}Z \) for surrounding lattices of all planar lattices. We will now show the link between this ice-type model and the loop model. This link is based on the fact that we can view each of the internal sites as a linear combination of two links. We do this by giving the lines in the following links an orientation

\[ \begin{array}{cc}
\text{ and } & \\
\end{array} \]

These correspond to the original links apart from a factor that is determined as follows. The oriented link differs a factor \( s \), with a power 1 if both lines turn right, a power \(-1\) if they both turn left and a power 0 if one line turns left and the other to the right. Then we map each internal sites to a linear combination of all the oriented links that match the orientation on the lines around the site. For example

\[ \begin{array}{cccc}
5 & \leftrightarrow & & \\
& & & \\
\end{array} \]

We now want the weights of the internal sites to correspond to the weights of the corresponding links. This gives us that the link

\[ \begin{array}{c}
\end{array} \]

has weight 1 and the link

\[ \begin{array}{c}
\end{array} \]

has weight \( x \). This gives us the connection between the icy-type model and a loop model and we have already seen the link between that and the Temperley-Lieb algebra. The only thing that is still missing is the precise link between the factor \( \beta = q + q^{-1} \) in the Temperley-Lieb algebra and the constants \( p, s \) and \( v \). This relation is as follows

\[ p = \beta^2, \quad s = \sqrt{q} \]

and \( \beta \) is also equal to the critical value of \( v \).
4.3. $O(n)$ model

The $O(n)$ model is a model that consist of $n$-dimensional spin vectors of fixed length such that the interactions do not change when all the spins are simultaneously rotated by elements in the $O(n)$ group.

We will now consider the $O(n)$ model on a square lattice. Here every edge of the lattice is associated with an $n$-dimensional unit vector. Then the Boltzmann weight is the product of local factors located round a vertex of the lattice,

$$e^{-\beta H} = \prod_{(i,j,k,l)} Q(s_i, s_j, s_k, s_l),$$

where the sites $i, j, k, l$ are ordered clockwise around the vertex. Then the partition function is given by

$$\int \left( \prod_i \frac{d s_i}{K_n} \right) \prod_{(i,j,k,l)} Q(s_i, s_j, s_k, s_l).$$

Where $K_n$ is the surface of the $n$-dimensional unit sphere. The local Boltzmann constant $Q$ is given by

$$Q(s_i, s_j, s_k, s_l) = 1 + u(s_1 \cdot s_2 + s_3 \cdot s_4 + s_2 \cdot s_3 + s_4 \cdot s_1) + v(s_1 \cdot s_3 + s_2 \cdot s_4)$$

$$+ w(s_1 \cdot s_2)(s_3 \cdot s_4) + (s_2 \cdot s_3)(s_4 \cdot s_1).$$

We can represent these local interaction via the following links

Here the link of type (1) has weight 1, the links of type (2) have weight $u$, the links of type (3) have weight $v$ and the links of type (4) have weight $w$. Then each term in the product over all the interaction factors $Q$ corresponds to a selection of these links for each interaction on the square lattice. Such a term in the expansion over the interaction factors $Q$ would have degree 0, 1 or 2 for each of the spin vectors. When the degree is one, the integration over that spin becomes

$$\int \frac{d s_i}{K_n} s_i = 0.$$  

We normalize the integration over the spins which results in

$$\int \frac{d s_i}{K_n} = 1$$
and finally we normalize the length of the spins which results in

\[ \int \frac{d s_i}{K_a} s_i^a s_i^b = \delta_{a,b}. \]

What this means in terms of the links is that all the non-zero terms in the expansion over the interaction factors \( Q \) correspond to a diagram where all the loops are closed. This is assuming that the spin is not on the boundary, if the spin is on the boundary the result depends on the boundary condition. For the links this result determines whether paths that end on the boundary are allowed. The weight for each term will only be non-zero when in the corresponding link model each loop is either closed or ends at the boundary.

Now we are interested in what the link models look like for various value of \( u, v \) and \( w \). It has been shown\(^2\) that these undergo phase transitions. We will now give a rough argument for why it is likely such a transition would occur when \( u = v = 0 \) and we change the value of \( w \).

The only possible link models that contribute a non-zero term when \( u = v = 0 \) will consist either completely of links of type 4 or of links of type 1. Then there will be one model that is completely empty with weight one and \( 2^N \) models each with weight \( w^N \), when there are \( N \) vertices on the grid. Then when we change the weight of \( w \), for low values of \( w \) we expect the completely empty model and when we increase the value of \( w \) we expect there to be a switch where the link model is completely filled. \( N \) tends to be very large so we expect this switch to be a sharp transition. This turns out to be a first-order phase transition, one of the most well-known examples of which is the transition between liquid water and water vapour.

When \( w = 0 \) and \( v = u \) a phase transition also occurs when changing \( v \) except that this is a second-order transition. Then we can give a sketch of the phase diagram where \( u = v \) and note that there must be a critical point between these two extremes where the transition goes from first-order to second-order.

![Phase Diagram](image)

This previous diagram was when \( u = v \), but we are now interested in the 3-dimensional phase diagram where we consider all three variables. Note that the previous phase diagram would correspond to the diagonal where \( u = v \). We will now consider some other axes of this 3-dimensional phase diagram in an attempt to determine what the full diagram might look like.

The first axis we are interested in is where \( w = 0 \). We already know that there exists a second-order transition somewhere on the line corresponding to \( u = v \) from the previous part.

When \( u \) is very small compared to \( v \) the actual lines we expect to see will most likely take the form of squares because these maximize the number of lines of type \((3)\) compared to the amount of lines of type \((2)\) and even if they are not exactly squares the amount of corners will be fairly low. The amount and size of these shapes will of course depend on the value of \( v \). Note that within each of these shapes it is very well possible to have smaller shapes and for high enough values of \( v \) this is very likely. A possible phase transition might occur when these various shapes start neighbouring one another, because for lower values of \( v \) we can expect the shapes to be far apart from each other.
When \( v \) is very small compared to \( u \), the lines we expect to see will be very erratic and only rarely will the line move straight. For low enough values of \( u \) this will mostly take the shape of small circles on only 4 squares. When \( u \) gets larger we would expect these shapes to increase in size and number. A possible phase transition might be when \( u \) gets large enough that we expect there to be a single line that covers most of the grid.
5. Conclusion

We have started of by examining the $n$-diagram algebra and found that it is equivalent to the Temperley-Lieb algebra, which is an algebra completely defined by its generators and their defining relations. We then began work on the representations of the $n$-diagram algebra. We found that the $n$-diagrams had a natural action on the $(n,k)$-link states. These were used to define the standard modules and using the degeneracy of a bilinear form on these standard modules we found that they were irreducible when $q$ was not a root of unity, $\beta = q + q^{-1}$ where $\beta$ is a variable for the $n$-diagram algebra. This lead to the result that the $n$-diagram algebra is semisimple and the standard modules form a complete set of irreducible modules when $q$ is not a root of unity.

Then we examine the dilute $n$-diagram algebra, which is a generalization of the $n$-diagram algebra. We start by giving a set of generators of the algebra and show that every dilute $n$-diagram can be constructed using these generators. We then define the dilute Temperley-Lieb algebra using these generators and a number of defining relations. Using these relations we show that every element in the dilute Temperley-Lieb algebra can be written as a linear combination of words in the same form as the one we used to construct all the dilute $n$-diagrams thus proving that the two algebras are in fact equivalent. Then we examine the representations of the dilute $n$-diagram algebra and similar to the regular $n$-diagram algebra we define the standard modules using dilute $n$-link states. Then we define a bilinear form on these standard modules and show using a few of the results on the degeneracy of the bilinear form on regular link states that these standard modules are irreducible. Which leads to the result that the dilute $n$-diagram is semisimple and that the standard modules form a complete set of irreducible modules when $q$ is not a root of unity.

We then give a short introduction to some of the basics of statistical physics. After which we define a two row transfer matrix for models for which states consist of paths on a square lattice. Then the link between this transfer matrix and the Temperley-Lieb algebra is shown. Next we introduce the Potts model and show that it is equivalent to an ice-type model on a square lattice. We show the link between this ice-type model and one which consists of paths on a square lattice thus giving us the link between the Potts model and the Temperley-Lieb algebra. Finally we examine the $O(n)$ model on a square lattice, we show that this corresponds to a model where the states consist of closed loops on a square lattice. Then we end the thesis with examining the possible phase transition that might occur when changing some of the variables in the $O(n)$ model.

A possibility for future research could be examining the representation of the regular and dilute $n$-diagram algebra for when $q$ is not a root of unity. For the regular $n$-diagram algebra this study can be found in [5] and for the dilute $n$-diagram algebra this can be found in [6].

Another possible avenue of research could be that of various generalizations of the Temperley-Lieb algebra such as the affine Temperley-Lieb algebra or various braid algebras.

Lastly there are a lot of applications of the dilute Temperley-Lieb algebra within statistical physics and in this thesis we have only scratched the surface of what can be done.
Bibliography


Populaire samenvatting

In dit project beschouwen we de wiskundige structuur achter de zogeheten *dilute n-diagrammen* en de link tussen deze dilute n-diagrammen en modellen uit de statistische fysica.

We beginnen nu met een uitleg geven over wat een *n*-diagram nu precies is. Zo’n diagram bestaat uit twee horizontale strepen boven elkaar met een gelijk aantal punten op iedere streep. Verder zijn er nog een aantal niet-kruisende lijnen die deze punten verbinden. Hierbij mag ieder punt met maximaal een lijn verbonden zijn, maar het is goed mogelijk dat een van de punten met geen enkele lijn verbonden is. In dit laatste geval geven we het punt weer met een kleine cirkel.

![Figure 5.1.](image)

**Figure 5.1.** Twee voorbeelden van diagrammen met vijf punten op iedere lijn.

Hierbij zeggen wij dat twee diagrammen hetzelfde zijn als het mogelijk is de een in de ander te veranderen door de lijnen te verbuigen, hierbij moeten de eindpunten wel hetzelfde blijven.

Nu kunnen we twee van deze diagrammen aan elkaar plakken door de een boven de andere te plaatsen en de middelste streep te verwijderen. Het resultaat hiervan is een nieuw *n*-diagram met hetzelfde aantal punten.

![Figure 5.2.](image)

**Figure 5.2.** De twee diagrammen uit figuur 5.1 zijn hier boven op elkaar geplakt, vervolgens is de middelste streep verwijderd en zijn de lijnen verbogen om een duidelijker diagram te krijgen.

Dit plakken geeft een wiskundige structuur en zorgt dat we kunnen werken met deze diagrammen. Dan vormen voor alle positieve gehele getallen *n* de diagrammen met *n* punten een zogeheten *algebra*. In deze scriptie hebben we een abstractere algebra gedefinieerd en laten zien dat de twee algebra’s hetzelfde zijn. Deze abstractere algebra heet de dilute Temperley-
Lieb algebra vernoemd naar Neville Temperley en Elliott Lieb die een vergelijkbare algebra in 1971 introduceerde om een aantal modellen in de statistische fysica op te lossen.

Statistische fysica gaat over het beschrijven van het gedrag van grote systemen aan de hand van hele kleine systemen. Hier zijn grote systemen over het algemeen systemen die we met het blote oog kunnen waarnemen zoals het koken van water, terwijl de hele kleine systemen gaan over het gedrag van enkele moleculen of atomen. Het probleem hier is dat zelfs als het gedrag van de hele kleine systemen bekend is, het heel moeilijk kan zijn of zelfs onmogelijk om het gedrag van het grote systeem te bepalen. Terugkomend op het voorbeeld van kokend water zijn er ongeveer $10^{26}$ moleculen in een liter water en $10^{26}$ berekeningen uitvoeren is zelfs voor een computer niet realistisch. Hierdoor zijn methodes die het rekenen aan verschillende modellen mogelijk maken erg belangrijk voor de statistische fysica.

Een van deze methodes is de dilute Temperley-Lieb algebra. Dit kan voor sommige modellen helpen met het berekenen van de partitiefunctie. Deze geeft veel informatie over het systeem op grote schaal terwijl berekenen alleen kennis vereist van het gedrag van het systeem op hele kleine schaal en staat dus erg centraal in de statistische fysica. Het berekenen van de partitiefunctie kan voor sommige modellen omgezet worden naar het herhaaldelijk aan elkaar plakken van de dilute $n$-diagrammen. In dit project is onderzocht hoe deze link tussen de diagrammen en het berekenen van de partitiefunctie werkt voor onder andere het Potts model. Dit model kan bijvoorbeeld gebruikt worden voor het modelleren van magneten.
A. Preliminaries

In this thesis we are interested in the representations of certain algebras. For an algebra \( A \) it is the case that a representation of \( A \) is the same as an \( A \)-module. We will now give a few results from representation theory that will be useful.

**Theorem A.1. (Wedderburn).** Let \( A \) be a complex, finite-dimensional, semisimple associative and \( M_1, \ldots, M_n \) a complete set of non-isomorphic irreducible modules. Then the regular module \( A \) decomposes as

\[
A \cong \bigoplus_{i=1}^{n} (\dim M_i) M_i.
\]

A proof for this theorem can be found in [13].

**Corollary A.2.** Let \( M_1, \ldots, M_n \) be irreducible and non-isomorphic left modules of some complex, finite dimensional, semisimple, associative algebra \( A \). These form a complete set of irreducible modules when

\[
\dim A = \sum_{i=1}^{n} (\dim M_i)^2.
\]

A.1. Tensor algebra

We will now introduce a construction that will allow us to define an algebra by generators and relations. This construction will allow us to obtain presentations of certain algebras in terms of generators and relations.

**Definition A.3.** Given a set \( \{ x_i \}_{i \in I} \) we will now define \( V \) to be the vector space over \( \{ x_i \}_{i \in I} \),

\[
V = \bigoplus_{i \in I} \mathbb{C} x_i.
\]

**Definition A.4.** Given a vector space \( V \) we define the tensor algebra \( T(V) \) to be,

\[
T(V) := \bigoplus_{m=0}^{\infty} V^\otimes m.
\]

This is the complex vector space with a linear basis given by the set \( \{ x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n} | i_k \in I, n \in \mathbb{Z}_{\geq 0} \} \). We define the multiplication on the basis to be:

\[
(x_{i_1} \otimes \cdots \otimes x_{i_1}) \cdot (x_{j_1} \otimes \cdots \otimes x_{j_k}) = x_{i_1} \otimes \cdots \otimes x_{i_1} \otimes x_{j_1} \otimes \cdots \otimes x_{j_k},
\]

where the word of length zero is the identity of this multiplication, extending this multiplication bilinearly turns \( T(V) \) into an algebra. We want to use this tensor algebra to give a more abstract presentation of some algebras in terms of generators and relations. For an algebra \( A \),
generated as an algebra by elements \{a_i\}_{i \in I} we have a surjective algebra map \( \phi : T(V) \to A \) defined by \( x_i \mapsto a_i \), where \( V = \bigoplus_{i \in I} \mathbb{C}x_i \). We now also have \( T(V)/\ker(\phi) \cong A \), we now want to describe \( \ker(\phi) \) as a two-sided ideal generated by explicit elements in \( \ker(\phi) \).

Given a number of polynomial relations \( p_\alpha((a_i)_{i \in I}) = 0 \) in \( A \), \( \alpha = 1, \ldots, m \), with the \( p_\alpha \) being polynomials where the non-commuting variables consist of the generating set \((a_i)_{i \in I}\), we can define \( I \subset T(V) \) to be the two-sided ideal generated by \( p_\alpha((x_i)_{i \in I}) = 0 \), \( \alpha = 1, \ldots, m \). Because these polynomial relations also hold in \( A \) we would have \( I \subset \ker(\phi) \). For example if in \( A \), \( a_1^2 + a_2 = 0 \), then the corresponding element in \( I \) would be \( x_1 x_1 + x_2 \).

**Definition A.5.** Given a set \( \{x_i\}_{i \in I} \) and a number of polynomial relations \( p_\alpha((x_i)_{i \in I}) = 0 \) on this set, \( \alpha = 1, \ldots, m \), if \( I \) is the two-sided ideal generated by these relations, then we call \( T(V)/I \) the algebra generated by \( \{x_i\}_{i \in I} \) with defining relations \( p_\alpha((x_i)_{i \in I}) \).

Note that we will often give relations in the form \( x_i^2 = x_j \). When we form the ideal generated by this relation we mean the ideal generated by \( x_i^2 - x_j = 0 \).

When we write elements of \( T(V)/I \) we will often neglect to write the tensor products. So we will write \( x_{i_1} \otimes \cdots \otimes x_{i_k} = x_{i_1} \cdots x_{i_k} \).

When \( A \) is an algebra and \( \{x_i\}_{i \in I} \) is a generating set for \( A \), for any set of polynomial relations that hold in \( A \) on \( \{x_i\}_{i \in I} \) we can define \( T(V)/I \). There will always be a surjective algebra homomorphism \( \phi : T(V)/I \to A \) where \( \phi(x_{i_1} \otimes \cdots \otimes x_{i_k}) = \phi(x_{i_1})\phi(x_{i_2})\cdots\phi(x_{i_k}) = x_{i_1}x_{i_2}\cdots x_{i_k} \). This algebra homomorphism is well defined because \( I \subset \ker(\phi) \).

Now our goal in this thesis is to give for some particular algebras, namely the regular \( n \)-diagram and the dilute \( n \)-diagram algebra, a natural set of algebraic generators \((a_i)_{i \in I}\) and a list of relations \( p_\alpha((a_i)_{i \in I}) = 0 \), \( \alpha = 1, \ldots, m \), such that the two sided ideal \( I \) generated by \( p_\alpha((a_i)_{i \in I}) - 0 \) equals \( \ker(\phi) \), where \( \phi : T(V) \to A \) is an algebra map such that \( \phi(x_i) = a_i \). In other words that \( A \cong T(V)/I \).